Solving the weighted stable set problem in claw-free graphs via decompositio

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We propose an algorithm for solving the maximum weighted stable set problem on claw-free graphs that runs in $\mathcal{O}(|V|(|E|+|V|\log|V|))-$time, drastically improving the previous best known complexity bound. This algorithm is based on a novel decomposition theorem for claw-free graphs, which is also introduced in the present paper. Despite being weaker than the structural results for claw-free graphs given by Chudnovsky and Seymour [2005], [2008], [2008] our decomposition theorem is, on the other hand, algorithmic, i.e., it is coupled with an $\mathcal{O}(|V||E|)-$time algorithm that actually produces the decomposition.

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1. INTRODUCTION

Given a graph $G(V, E)$, a *matching* is a set of non incident edges of E and a *stable set* is a set of pairwise non adjacent vertices of V. Edmonds [1965] proved that the weighted matching problem can be solved in $O(|V|^4)$-time for any graph. This worst case complexity was later improved by other authors, the best bound currently being $O(|V|(|V|log|V| + |E|)$ [Gabow 1990]. (Unless otherwise specified, graphs in this paper are undirected and simple. Sometimes, for a graph G, we let $V(G)$ and $E(G)$ denote respectively the vertex set and the edge set.)

Given a (multi-)graph G, one defines the *line graph* H of G as the intersection graph of the edges of G. G is called a *root graph* of H. A graph H is then said to be *line* if it is the line graph of some (multi-)graph G. There is a one-to-one correspondence between matchings in G and stable sets in H. Therefore, since a root graph G of a line graph H can be computed efficiently (this can be done in $O(max\{|E(H)|, |V(H)|\}$)-time, see [Roussopoulos 1973] for line graphs of simple graphs and [King 2009, pp 67-68] for line graphs of multi-graphs), a maximum weighted stable set (MWSS, in the following) in H can be found in time $O(|V(H)|^2 log(|V(H)|))$ (observe that the root graph will have $|V(H)|$ edges and $O(|V(H)|)$ vertices).

Line graphs have the property that the neighborhood of any vertex can be covered by two cliques, and the graphs with this latter property are called *quasi-line graphs*. A *claw* $\{u; s_1, s_2, s_3\}$ is the graph with vertices u, s_1, s_2, s_3 and edges us_i for $i = 1, 2, 3$; a graph is *claw-free* if no induced subgraph of G is isomorphic to a claw, i.e., if no vertex has a stable set of size three in its neighborhood. Claw-free graphs thus generalize quasi-line graphs, that in their turn generalize line graphs. Interestingly, the crucial augmenting path property of matchings extends to stable sets in claw-free graphs: a stable set of a claw-free graph is of maximum size if and only if there are no augmenting paths with respect to it (a path P is *augmenting* with respect to a stable set S if $(V(P)\setminus S)\cup(S\setminus V(P))$ is a stable set of size $|S|$+1). In fact, while the stable set problem is $\mathcal{NP}$-hard in general, it was proven it can be solved in polynomial time for claw-free graphs: Sbihi [1980] and later Lovász and Plummer [1986] gave algorithms for the cardinality case, while Minty [1980] solved the weighted version. The Minty algorithm was revised by Nakamura and Tamura [2001] and later simplified

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by Schrijver [2003] and can be implemented to run in time $O(|V|^6)$ for a claw-free graph $G(V, E)$. However, recently Nobili and Sassano [2010] could build upon the main ideas of Minty's algorithm and solve the problem in time $O(|V|^4 log(|V|)$, a significant improvement.

A deep breakthrough in our understanding of the structure of claw-free graphs is due to the recent seminal work of Chudnovsky and Seymour, see e.g. [2008], [2008]. The theory is too complex to describe in detail here (a good starting point is [Chudnovsky and Seymour 2005] where the authors overview their series of papers). In particular, a deep decomposition theorem for claw-free graphs is given in [Chudnovsky and Seymour 2008]. Inspired by those results, Oriolo, Pietropaoli and Stauffer [2008] proposed a new approach to solve the MWSS problem on graphs that admit a suitable decomposition, when this decomposition is *given* (in the same paper, the authors also developed a $O(|V|^6)$-time algorithm for the problem in claw-free graphs, by means of some graph reductions and an algorithmic decomposition theorem for a subclass of quasi-line graphs). Note that, in order to combine the results in [Chudnovsky and Seymour 2008] and [Oriolo et al. 2008] together, as to get an algorithm to solve the MWSS problem in claw-free graphs, one needs a (polynomial time) algorithm to get the decomposition in [Chudnovsky and Seymour 2008]. Until very recently no such algorithm was available; however, while preparing this paper, we knew that a lighter version of the result in [Chudnovsky and Seymour 2008] has been algorithmized by Hermelin, Mnich, van Leeuwen and Woeginger [2011].

In this paper, we provide a new decomposition theorem for claw-free graphs and a $\mathcal{O}(|V||E|)$-time algorithm to actually obtain the decomposition. Our theorem is inspired by ideas and tools developed by Chudnovsky and Seymour (as well as by the weaker decomposition theorem in [Oriolo et al. 2008]), but it is a stand-alone result that, even if less detailed than their decomposition theorem, is particularly useful when dealing with the MWSS problem. In fact, building upon our novel decomposition theorem, a few algorithmic results from the literature, and following the approach in [Oriolo et al. 2008] for finding a MWSS on graphs that admit a suitable decomposition, we show that we can solve the MWSS problem in a claw-free graph $G(V, E)$ in $\mathcal{O}(|V|(|E| + |V| \log |V|))-$time. This is to the best of our knowledge the fastest algorithm to solve the problem and this improves drastically upon previous known algorithms. Moreover it almost closes the algorithmic gap between stable set in claw-free graphs and matching (in fact, as observed earlier, the MWSS problem in a line graph $G(V, E)$ can be solved in $O(|V|^2 log(|V|))$-time).

We emphasize that our algorithms are *simple* and that they *can be implemented*: in fact a simplified version of our MWSS algorithm was implemented recently (and tested) in python by Dubois and Rouire, two first year master students at the University of Bordeaux ; Dubois, Rouire and Stauffer are planing to contribute this algorithm to networkx, a python package for graph structures and algorithms. Also, in order to reach the aforementioned time-complexity, they only use elementary data structures. Finally, we believe that our algorithmic decomposition result is interesting on its own and might be also useful to solve other kind of problems on claw-free graphs.

1.1. A bird's-eye view of the paper

In the following, we discuss how the paper is organized and give a few more details about the main results in it, and the way they relate to other results in the literature.

Section 2 is devoted to the definition of strips and their basic properties, while Section 2.1 deals with the MWSS problem in graphs that are the composition of strips. By now, it is enough to recall that a strip is a graph with either one or two designated cliques, called extremities, and that in order to compose some set of strips we have first to take the disjoint union of the strips, and then make some extremities pairwise complete to each other, i.e., "glue" them together. As we discuss in Section 2, the composition of strips can be seen as a generalization of a procedure to build line graphs; in fact, the main result of Section 2.1 is Theorem 2.10 showing that, if we are interested in solving the MWSS problem on a (general) graph G that is the composition of some *given* set of strips, and we are able to solve the MWSS problem on each strip, then a MWSS of G can be found by solving a single matching problem .

From Section 3 on, we will focus on claw-free graphs. In order to find a strip decomposition of a claw-free graph, we have to somehow revert the composition procedure described above, and, therefore, find some suitable clique to "unglue": this will require a careful analysis of the neighborhood of each vertex. Let v be a vertex of a claw-free graph G: if the closed neighborhood of v, i.e., $N(v) \cup \{v\}$, can be covered by two cliques K_1, K_2, then v is called *regular*, and, in case this covering is unique, K_1 and K_2 are *crucial* for v. An *articulation cliques* is a maximal clique K that is crucial for each of its vertices. We will show that every claw-free graph G, that has some articulation clique, admits a strip decomposition. When G is quasi-line things are much easier as, in order to get this decomposition, it is sufficient to *simultaneously* unglue *all* articulation cliques, while, when G is not quasi-line, i.e., there are some vertices that are non-regular, we first have to remove a strip "around" each non-regular vertex, as to end up with a quasi-line graph, where we can proceed as in the former case.

We will in fact have two slightly different decomposition theorems, one for quasi-line graphs and one for claw-free non quasi-line graphs (one can easily combine them as to get a decomposition theorem for all claw-free graphs). The main decomposition result for quasi-line graphs is Theorem 4.8, for which we give here a lighter statement:

THEOREM 1.1. *Let $G(V, E)$ be a connected quasi-line graph. There exists an algorithm that in time $O(|V||E|)$:*

(j) either recognizes that G is net-free;
(jj) or provides a strip-decomposition of G, such that, for each strip, the graph in the strip is distance simplicial with respect to its extremities.

(See Definition 1.6 for net-free graphs and Definition 1.5 for graphs that are distance simplicial with respect to some clique.) We will in fact show (Lemma 3.10) that a claw-free graph without articulation cliques is net-free. Following Theorem 1.1, in order to solve the MWSS problem in a quasi-line graph, we delve into two cases. In Section 3.1 we deal with the case where $G(V, E)$ is $\{$claw, net$\}$-free (and therefore quasi-line and net-free); in this case, we will provide an $O(|V||E|)$-time algorithm that builds upon two results from the literature: an $O(|V|^4)$-time algorithm to solve the maximum stable set problem in the *weighted* case by Pulleyblank and Shepherd [1993], and an $O(|V|^3)$-time algorithm for the *cardinality* case by Brandstadt and Dragan [2003]. In Section 4.1 we deal with the case where G has a strip decomposition as in part (jj) of Theorem 1.1; there, following Theorem 2.10, it is enough to show how to solve the MWSS problem in a graph that is distance simplicial with respect some clique: to this purpose the algorithm in [Pulleyblank and Shepherd 1993] suffices.

We point out that Theorem 1.1 is quite similar to the following theorem by Chudnovsky and Seymour (we skip the definitions of fuzzy circular interval graph and fuzzy linear interval strips, as we do not need them in the following):

THEOREM 1.2. *[Chudnovsky and Seymour 2005] Let $G(V, E)$ be a connected quasi-line graph.*

(j) Either G is a fuzzy circular interval graph;
(jj) or G is the composition of fuzzy linear interval strips.

So far we have not investigated which are the relationships between the two theorems. However, we point out that Theorem 1.1 is an algorithmic theorem, while Theorem 1.2 is not. On the other hand, while Theorem 1.2 could be used to prove the Ben Rebea conjecture [Eisenbrand et al. 2008], we do not know how to use Theorem 1.1 to get the same result; so this might suggest that Theorem 1.2 gives more details about the structure of quasi-line graphs.

The main decomposition theorem for claw-free graphs is Theorem 5.6, for which we again offer here a lighter statement:

THEOREM 1.3. *Let $G(V, E)$ be a connected claw-free but not quasi-line graph. There exists an algorithm that in time $O(|V||E|)$:*

(i) either recognizes that $\alpha(G) \leq 3$;
(ii) or provides a strip-decomposition of G, such that, for each strip, the graph in it is:
— either non quasi-line and with stability number at most 3;
— or distance simplicial with respect the extremities of the strip.

(The *stability number* of a graph is the maximum size of a stable set). Chudnovsky and Seymour also offer a decomposition theorem for claw-free graphs, that for our purposes is enough to describe as follows:

THEOREM 1.4. *[Chudnovsky and Seymour 2008] Let $G(V, E)$ be a connected claw-free but not quasi-line graph.*

(j) Either $\alpha(G) \leq 3$ and G belongs to a small set of basic graphs;
(jj) or G is the composition of strips that are:
— either from a small number of basic classes of strips, each such that the graph in it is non-quasi-line and has stability number at most 3;
— or fuzzy linear interval strips.

In fact, the structure of claw-free graphs and strips that have small stability number is described in detail by Chudnovsky and Seymour. That is not the case with Theorem 1.3; on the other hand, our main motivation is the solution of the MWSS problem, and with respect to that aim Theorem

1.3 suffices. In fact, when $\alpha(G) \leq 3$, we can compute a MWSS by enumeration. When G has a strip decomposition as in part (ii) of Theorem 1.3, following Theorem 2.10, it is enough to show how to solve the MWSS problem in a graph that is distance simplicial with respect some clique (see above) and in a graph with small stability number (enumeration).

Before moving to settling some more definitions, we conclude this section with a couple of computational remarks. Recall that, for a graph G on n vertices, we can obtain in $O(n^2)-$time a representation of G via adjacency lists from a representation of G via adjacency matrix, and vice versa. Often, this $O(n^2)$ extra time will not affect the global running time. So, unless differently stated, we shall assume that the graphs we deal with (and in particular the input graph) are stored via either representations. Last, it will be sometimes convenient to assume that we are given some linear order on the set of vertices of the input graph (e.g. the one induced when the graph is stored).

1.2. Definitions

For a non-negative integer k, we let $[k]$ denote the set $\{1, 2, \ldots, k\}$.

Let $G(V, E)$ be a graph. The complement of G is denoted by $\overline{G}$, while $G[S]$ denotes the subgraph induced by a set $S \subseteq V$. If $S \subseteq V$, we let $G \setminus S := G[V \setminus S]$. A *clique* is a set of pairwise adjacent vertices. We denote by $\alpha(G)$ ($\alpha_w(G)$) the maximum size (resp. weighted with respect to $w : V \mapsto \mathbb{R}$) stable set in G, and, for a set $S \subseteq V$, we let $\alpha(S) = \alpha(G[S])$ (resp. $\alpha_w(S) = \alpha_w(G[S])$).

We denote by $N(v)$ the *open neighborhood* of a vertex $v \in V$, i.e., the set of vertices that are adjacent to v; we let $N[v] = N(v) \cup \{v\}$ be the *closed neighborhood*. For a set $S \subseteq V$ we let $N(S) := \bigcup_{v \in S} N(v) \setminus S$ and $N[S] := \bigcup_{v \in S} N[v]$. A vertex u is *universal* to v if $N[v] \subseteq N[u]$ and we let $U(v)$ be the set of vertices that are universal to v and let $U[v] \cup \{v\}$. Two vertices u and v that are universal to each other are *true twins*. We also denote by $N_j(v)$ the set of vertices that are at distance j (in terms of number of edges) from v (therefore $N_1(v) = N(v)$). A vertex $v \in V$ is *simplicial* if $N(v)$ is a clique, and we denote by $Simp(G)$ the set of simplicial vertices of G.

Definition 1.5. We say that a clique K of a connected graph G is *distance simplicial* if, for every j, $\alpha(N_j(K)) \leq 1$. In this case, we also say that G is distance simplicial with respect to K.

Definition 1.6. A *net* $\{v_1, v_2, v_3; s_1, s_2, s_3\}$ is the graph with vertices $v_1, v_2, v_3, s_1, s_2, s_3$ and edges v_1v_2, v_1v_3, v_2v_3, and v_is_i for $i = 1, 2, 3$. v_1, v_2, v_3 is called the *triangle* of the net. We say that G is *net-free* if no induced subgraph of G is isomorphic to a net and call *net clique* every maximal clique of G that contains the triangle of a net.

Let $k \geq 3$. A $k-$*hole* is a chordless cycle with k vertices. A k-*anti-hole* is the complement of a $k-$hole. A $k-$*wheel* is a graph with vertex set $\{v\} \cup C$, where C induces a k-hole and v is complete to C: v is the *center* of the wheel. Analogously, a $k-$*anti-wheel* is a graph with vertex set $\{v\} \cup C$, where C induces a k-anti-hole, and v is complete to C: v is the *center* of the anti-wheel. A $k-$hole (resp. k-anti-hole, k-wheel, k-antiwheel) is *odd* if k is odd. We say that a k-anti-wheel is *long* if $k > 5$.

2. STRIPS AND THE STABLE SET PROBLEM

Chudnovsky and Seymour [2005] introduced a composition operation in order to define their structural results for claw-free graphs. This composition operation is general and applies to non-claw-free graphs as well. In order to better grasp this operation, we first deal with an algorithmic procedure that can be used to build line graphs.

Given a graph G, each vertex in G is associated with a clique in the line graph $H = L(G)$ (all edges incident to this vertex are pairwise adjacent in H). If we let $\mathcal{F}$ denote the family of cliques of H that are associated with vertices of G, we observe that $\mathcal{F}$ has the following properties: (i) every edge of H is covered by some clique of $\mathcal{F}$; (ii) every vertex of H is covered by exactly two cliques of $\mathcal{F}$. Krausz [1943] proved the following:

LEMMA 2.1. *[Krausz 1943] A graph $G(V, E)$ is the line graph of a multi-graph if and only if there exists a family of cliques $\mathcal{F}$ such that every edge in E is covered by a clique from the family, and moreover every vertex in V is covered by at most two cliques from the family.*

This theorem gives an algorithmic procedure to build line graphs. This procedure requires as input a set of vertices V and a partition $\mathcal{P} = P_1, ..., P_q$ of the multi-set $V \cup V$. It then associates to the pair $(V, \mathcal{P})$ the graph G with vertex set V and edge set $E := \{\{u, v\} : u \neq v$ and both $u, v \in P_i$, for some $1 \leq i \leq q\}$. Chudnovsky and Seymour generalized the above construction, essentially by replacing *vertices* with *strips*. We borrow (but slightly change) some definitions of theirs.

Definition 2.2. A *strip* $(G, \mathcal{A})$ is a graph G (not necessarily connected) with a multi-family $\mathcal{A}$ of either one or two designated non-empty cliques of G.

If $\mathcal{A}$ is made of a single clique, then $(G, \mathcal{A})$ is a *1-strip*, if $\mathcal{A}$ is made of two cliques A_1, A_2, then $(G, \mathcal{A})$ is a *2-strip* (in this case, possibly $A_1 = A_2$ since $\mathcal{A}$ is a multi-family). The cliques in $\mathcal{A}$ are called the *extremities* of the strip, while the *core* $C(G, \mathcal{A})$ of the strip is made of the vertices that do not belong to the extremities.

Let $\mathcal{G} = \{(G^1, \mathcal{A}^1), \ldots, (G^k, \mathcal{A}^k)\}$ be a family of k vertex disjoint strips; we can *compose* the strips in $\mathcal{G}$, according to the operation we define below. Note that we denote by $\bigcup_{j\in[k]} \mathcal{A}^j$ the *multi-family* whose elements are the *extremities* from each $\mathcal{A}^j$: again, it is a multi-family, as the two extremities of a same strip need not to be different.

Definition 2.3. Let $\mathcal{F} = \{(G^1, \mathcal{A}^1), \ldots, (G^k, \mathcal{A}^k)\}$ be a family of k vertex disjoint strips and let $\mathcal{P} := \{P_1, ..., P_m\}$ be a partition of the multi-family of the extremities $\bigcup_{j\in[k]} \mathcal{A}^j$. The *composition of the family of strips $\mathcal{F}$ with respect to the partition $\mathcal{P}$* is the graph G such that:

— $V(G) = \bigcup_{j=1}^{k} V(G^j)$;
— two vertices $u, v \in V(G)$ are adjacent if and only if either $u, v \in V(G^j)$ and $\{u, v\} \in E(G^j)$, for some $j \in [k]$, or there exist $A \in \mathcal{A}^i$ and $A' \in \mathcal{A}^j$, for some $1 \le i \le j \le k$, such that $u \in A$, $v \in A'$, and A and A' are in the same class of $\mathcal{P}$.

In this case, we say that $(\mathcal{F}, \mathcal{P})$ defines a *strip decomposition* of G. Note also that, for each class $P \in \mathcal{P}$, the set of vertices $\bigcup_{A\in P} A$ is a clique of G, that is called a *partition-clique*.

In the following, when we say that G is the composition of some family of strips with respect to some partition $\mathcal{P}$, we assume that the strips are vertex disjoint and that $\mathcal{P}$ gives a partition of the multi-family of the extremities of these strips. We skip the straightforward proof of the next lemma:

LEMMA 2.4. *Let G be the composition of a family of strips $\{(G^1, \mathcal{A}^1), \ldots, (G^k, \mathcal{A}^k)\}$, with respect to some partition $\mathcal{P}$. Then the following statements hold:*

— *for each $j \in [k]$, the core $C(G^j, \mathcal{A}^j)$ of the strip $(G^j, \mathcal{A}^j)$ is anti-complete to $V(G) \setminus V(G^j)$ and $G[C(G^j, \mathcal{A}^j)] = G^j[C(G^j, \mathcal{A}^j)]$;*
— *for each $j \in [k]$, $G[V(G^j)] = G^j$ if either G^j is a 1-strip, or it is a 2-strip and its extremities belong to different classes of $\mathcal{P}$; else $G[V(G^j)]$ is obtained from G^j making the extremities in $\mathcal{A}^j$ complete to each other.*
— *each edge between different strips G^i and G^j is an edge between their extremities and is induced by some partition-clique.*

One can easily build a graph G that is the composition of strips $\{(G^j, \mathcal{A}^j), j \in [k]\}$ such that each G^j is claw-free/quasi-line/line but G itself is not claw-free/quasi-line/line. However, this is not possible, as soon as we require that, for each strip, the property we are interested in (claw-freeness/quasi-lineness/lineness) holds on an *auxiliary* graph that we associate to the strip. This leads to the following:

Definition 2.5. We say that a strip $(G, \mathcal{A})$ is claw-free/quasi-line/line if the graph G_+ that is obtained from G as follows:

— if G is a 2-strip, with $\mathcal{A} = \{A_1, A_2\}$, add two additional vertices a_1, a_2 such that $N(a_i) = A_i$, for $i = 1, 2$;
— if G is a 1-strip, with $\mathcal{A} = \{A_1\}$, add one additional vertex a_1 such that $N(a_1) = A_1$,

is claw-free/quasi-line/line.

We skip the proof of the following simple lemma.

LEMMA 2.6. *The composition of claw-free / quasi-line strips is a claw-free / quasi-line graph.*

LEMMA 2.7. *Let G be the composition of a family of k line strips $\{(G^j, \mathcal{A}^j), j \in [k]\}$ with respect to a partition $\mathcal{P}$. Then G is a line graph.*

PROOF. From Lemma 2.1, we know that the strips G^j are line if and only if, for all $j = 1, ..., k$, there exists a set of cliques $\mathcal{F}^j$ of G^j_+ such that: every edge from G^j_+ is covered by a clique of $\mathcal{F}^j$; each vertex in G^j_+ is covered by at most two cliques of $\mathcal{F}^j$. In fact, we may assume without loss of generality that the set $\mathcal{F}^j$ is also such that the vertices from $V(G^j_+) \setminus V(G^j)$, that are simplicial, are covered by exactly one clique of $\mathcal{F}^j$ (if a vertex v of $V(G^j_+) \setminus V(G^j)$ is covered by two cliques F_1, F_2, then we can slightly change $\mathcal{F}^j$ into $\mathcal{F}^j \setminus \{F_1, F_2\} \cup \{N[v]\}$). We denote by F^j the set of cliques of $\mathcal{F}^j$ covering the vertices from $V(G^j_+) \setminus V(G^j)$ (F^j is of cardinality one if $(G^j, \mathcal{A}^j)$ is a 1-strip and two if $(G^j, \mathcal{A}^j)$ is a 2-strip). Let $\widetilde{\mathcal{F}}^j := \mathcal{F}^j \setminus F^j$. Consider the family of cliques $\widetilde{\mathcal{F}}$ of G made of the union of

$\widetilde{\mathcal{F}}^j$ for all j and the partition-cliques defined by $\mathcal{P}$. By definition of composition, $\widetilde{\mathcal{F}}$ covers all edges in G and moreover every vertex in G is covered by at most two cliques. The result follows then again from Lemma 2.1. □

(We point out that one might impose properties on the strips $\{(G^j, \mathcal{A}^j), j \in [k]\}$ in order to avoid using the artifact of additional vertices, and still get an analogous of Lemma 2.6 and Lemma 2.7, see [Chudnovsky and Seymour 2008]: for our purpose, this unnecessarily complicates the exposition.)

2.1. The maximum weighted stable set problem in composition of strips

In this section, we show that we can solve, in polynomial time, the maximum weighted stable set problem in a graph G that is the composition of strips, as soon as we are able to solve in polytime the same problem on each strip of G. The main tools are Lemma 2.7 and a simple reduction that replaces each strip with a simple line strip.

More precisely, let $G(V, E)$ be the composition of k strips $H_1 = (G^1, \mathcal{A}^1), \dots, H_k = (G^k, \mathcal{A}^k)$, with respect to a partition $\mathcal{P}$ and let $w : V(G) \mapsto \mathbb{R}$. We show that we can replace each strip H_i with a simple *line* strip H_i' and reduce the mwss problem on G to the same problem on a graph G^*, that is the composition of the strips $H_1', H_2', \dots, H_k'$ with respect to a suitable partition $\mathcal{P}^*$, where we define a suitable weight function $w^* : V(G^*) \mapsto \mathbb{R}$. As G^* is line from Lemma 2.7 and we can build a root graph easily, we can find a MWSS by solving a matching problem.

The rationale in replacing a strip, say H_1, with another strip H_1' is the following. The only possible obstruction to combine a stable set T of $G \setminus V(G^1)$ and a stable set U of G^1 into a stable set of G are the adjacencies in the partition-cliques involving the extremities of H_1. Because those extremities are cliques, there are four possible configurations describing the interactions between U and the extremities of H_1 (by now assume that H_1 is a 2-strip): U contains a vertex in both extremities; U contains a vertex in one or the other extremity; U does not contain any vertex in the extremities. When one is interested in a MWSS of G then, given the stable set T for $G \setminus V(G^1)$, one obviously wants the stable set U to be of maximum weight among the stable sets from configurations that are *compatible* with T. Hence, we can replace H_1 with another strip H_1' as long as they agree, *for each configuration*, on the value of a MWSS.

The strip H_1' will be fairly trivial: it will be either the strip $H_1' = (C_1, \{c_1\})$ or the strip $H_1' = (C_3, \{\{c_1, c_3\}, \{c_2, c_3\}\})$, where we denote by C_k the complete graph on $k \geq 1$ vertices labeled $c_1, ..., c_k$ (hence, C_1 is the graph made of a single vertex, while C_3 is a triangle). However, because the composition, and thus the adjacencies between $G \setminus V(G^1)$ and G^1, are slightly different if: (i) H_1 is a 1-strip; (ii) H_1 is a 2-strip and its extremities are in the same class of the partition $\mathcal{P}$; or (iii) H_1 is a 2-strip and its extremities are in different classes of the partition $\mathcal{P}$, we need to distinguish those cases. In each case, we define $w'(v) = w(v)$ for $v \notin V(G^1)$.

— In case (i), i.e., when $\mathcal{A}^1 = \{A_1\}$ and there exists $P \in \mathcal{P} : A_1 \in P$, we define: $H_1' = (C_1, \{c_1\})$; $\delta_1 = \alpha_w(G^1 \setminus A_1)$; $w'(c_1) = \alpha_w(G^1) - \delta_1$; $\mathcal{P}' := (\mathcal{P} \setminus P) \cup (P \cup \{c_1\} \setminus A_1)$.

— In case (ii), i.e., when $\mathcal{A}^1 = \{A_1, A_2\}$ and there exists $P \in \mathcal{P} : A_1, A_2 \subseteq P$, we define: $H_1' = (C_1, \{c_1\})$; $\delta_1 = \alpha_w(G^1 \setminus (A_1 \cup A_2))$; $w'(c_1) = \max\{\alpha_w(G^1 \setminus A_1), \alpha_w(G^1 \setminus A_2), \alpha_w(G^1 \setminus A_1 \Delta A_2)\} - \delta_1$; $\mathcal{P}' := (\mathcal{P} \setminus P) \cup (P \cup \{c_1\} \setminus \{A_1, A_2\})$.

— In case (iii), i.e., when $\mathcal{A}^1 = \{A_1, A_2\}$ and there exist $P_1 \neq P_2 \in \mathcal{P} : A_i \in P_i$ $i = 1, 2)$, we define: $H_1' = (C_3, \{\{c_1, c_3\}, \{c_2, c_3\}\})$; $\delta_1 = \alpha_w(G^1 \setminus (A_1 \cup A_2))$; $w'(c_1) = \alpha_w(G^1 \setminus A_2) - \delta_1$, $w'(c_2) = \alpha_w(G^1 \setminus A_1) - \delta_1$ and $w'(c_3) = \alpha_w(G^1) - \delta_1$; $\mathcal{P}' := (\mathcal{P} \setminus (P_1 \cup P_2)) \cup ((P_1 \setminus A_1) \cup \{c_1, c_3\}) \cup ((P_2 \setminus A_2) \cup \{c_2, c_3\})$.

The next lemma follows easily from the above discussion.

LEMMA 2.8. *Let G be the composition of k strips $H_1 = (G^1, \mathcal{A}^1), \dots, H_k = (G^k, \mathcal{A}^k)$, with respect to a partition $\mathcal{P}$ and let $w : V(G) \mapsto \mathbb{R}$. Let G' be the composition of $H_1', H_2, \dots, H_k$ with respect to the partition $\mathcal{P}'$ and $w' : V(G') \mapsto \mathbb{R}$, with H_1', $\mathcal{P}'$ and w' defined above. Then $\alpha_w(G) - \delta_1 = \alpha_{w'}(G')$. Moreover any MWSS of G' (with respect to w') can be converted into a MWSS of G (with respect to w) if the following stable sets are known: a MWSS of G^1; a MWSS of G^1 not intersecting A, for each $A \in \mathcal{A}^1$; a MWSS of G^1 not intersecting $\bigcup_{A \in \mathcal{A}^1} A$; a MWSS of G^1 not intersecting $A_1 \Delta A_2$ (this one is required only if $\mathcal{A}^1 = \{A_1, A_2\}$ and A_1, A_2 are in the same class of $\mathcal{P}$).*

PROOF. (i) We begin with showing that $\alpha_w(G) \leq \delta_1 + \alpha_{w'}(G')$. Let S be a MWSS of G. First suppose that S picks a vertex in A_1. Then $S \cap V(G^1)$ is a mwss in G^1 (otherwise we would swap with a better one in G^1). Also S is not picking any vertex belonging to an extremity in P other than A_1, and therefore $S' = (S \setminus V(G^1)) \cup \{c_1\}$ is a stable set of G'. Therefore, $\alpha_w(G) = w(S) = w'(S') - w'(c_1) + w(S \cap V(G^1)) = w'(S') - w'(c_1) + \alpha_w(G^1) = w'(S') + \delta_1 \leq \alpha_{w'}(G') + \delta_1$. Suppose now that S does not pick any vertex from A_1. Then $S \cap V(G^1)$ is a mwss in $G^1 \setminus A_1$, and $S \setminus V(G^1)$ is a stable set of G'. Therefore, $\alpha_w(G) = w(S) = w(S \cap V(G^1)) + w(S \setminus V(G^1)) \leq \delta_1 + \alpha_{w'}(G')$.

We now show that $\alpha_w(G) \geq \delta_1 + \alpha_{w'}(G')$. Let S' be a MWSS of G'. First suppose that S' picks c_1. In this case, for any stable set S of G^1, $(S' \setminus c_1) \cup S$ is a stable set of G. Therefore, if in particular we choose S as a mwss of G^1, $\alpha_w(G) \geq w((S' \setminus c_1) \cup S) = w'(S') - w'(c_1) + \alpha_w(G^1) = \alpha_{w'}(G') + \delta_1$. Now suppose that S' does not pick c_1. In this case, for any stable set S of $G^1 \setminus A_1$, $S' \cup S$ is a stable set of G. Therefore, if in particular we choose S as a mwss of $G^1 \setminus A_1$, $\alpha_w(G) \geq w(S' \cup S) = w'(S') + \alpha_w(G^1 \setminus A_1) = \alpha_{w'}(G') + \delta_1$.

Therefore, $\alpha_w(G) = \delta_1 + \alpha_{w'}(G')$. Moreover, if S' is a MWSS of G', we may derive from it a MWSS of G, as soon as we are given: a MWSS of G^1; a MWSS of G^1 not intersecting A_1.

(ii). This case easily reduces to the previous one. In fact, let $\overline{G}^1$ be the graph obtained from G^1 making A_1 complete to A_2, $\overline{H_1}$ be the 1-strip $(\overline{G}^1, A_1 \cup A_2)$, and finally $\overline{\mathcal{P}}$ be the partition obtained from $\mathcal{P}$ by replacing P with $P \cup \{A_1 \cup A_2\} \setminus \{A_1, A_2\}$. Then G is the composition of $\overline{H}_1, H_2, \ldots, H_k$ with respect to the partition $\overline{\mathcal{P}}$. Now the statement follows from the previous case, as soon as we observe that $\alpha_w(\overline{G}^1) = \max\{\alpha_w(G^1 \setminus A_1), \alpha_w(G^1 \setminus A_2), \alpha_w(G^1 \setminus A_1 \Delta A_2)\}$ and $\alpha_w(\overline{G}^1 \setminus (A_1 \cup A_2)) = \alpha_w(G^1 \setminus (A_1 \cup A_2))$.

(iii). Let S be a MWSS of G. First suppose that S intersects both A_1 and A_2. Then $S \cap V(G^1)$ is a MWSS in G^1. Also $S' = (S \setminus V(G^1)) \cup \{c_3\}$ is a stable set of G'. Therefore, $\alpha_w(G) = w(S) = w'(S') - w'(c_3) + w(S \cap V(G^1)) = w'(S') - w'(c_3) + \alpha_w(G^1) = w'(S') + \delta_1 \leq \alpha_{w'}(G') + \delta_1$. Suppose now that S picks a vertex in A_1 but no vertex in A_2. Then $S \cap V(G^1)$ is a MWSS in $G^1 \setminus A_2$. Also $S' = (S \setminus V(G^1)) \cup \{c_1\}$ is a stable set of G'. Therefore, $\alpha_w(G) = w(S) = w'(S') - w'(c_1) + w(S \cap V(G^1)) = w'(S') - w'(c_1) + \alpha_w(G^1 \setminus A_2) = w'(S') + \delta_1 \leq \alpha_{w'}(G') + \delta_1$. The case where S picks a vertex in A_2 but no vertex in A_1 goes along the same lines. Finally suppose now that S does not pick any vertex from $A_1 \cup A_2$. Then $S \cap V(G^1)$ is a MWSS in $G^1 \setminus (A_1 \cup A_2)$, while $S \setminus V(G^1)$ is a stable set of G'. Therefore, $\alpha_w(G) = w(S) = w(S \cap V(G^1)) + w(S \setminus V(G^1)) \leq \delta_1 + \alpha_{w'}(G')$.

Conversely, let S' be a MWSS of G'. First suppose that S' picks c_3. In this case, for any stable set S of G^1, $(S' \setminus c_3) \cup S$ is a stable set of G. Therefore, if in particular we choose S as a MWSS of G^1, $\alpha_w(G) \geq w((S' \setminus c_3) \cup S) = w'(S') - w'(c_3) + \alpha_w(G^1) = \alpha_{w'}(G') + \delta_1$. Now suppose that S' picks c_1. In this case, for any stable set S of $G^1 \setminus A_2$, $(S' \setminus c_1) \cup S$ is a stable set of G. Therefore, if in particular we choose S as a MWSS of $G^1 \setminus A_2$, $\alpha_w(G) \geq w((S' \setminus c_1) \cup S) = w'(S') - w'(c_1) + \alpha_w(G^1 \setminus A_2) = \alpha_{w'}(G') + \delta_1$. The case where S' picks c_2 goes along the same lines. Finally suppose that S' does not pick any vertex from C_3. In this case, for any stable set S of $G^1 \setminus (A_1 \cup A_2)$, $S' \cup S$ is a stable set of G. Therefore, if in particular we choose S as a MWSS of $G^1 \setminus (A_1 \cup A_2)$, $\alpha_w(G) \geq w(S' \cup S) = w'(S') + \alpha_w(G^1 \setminus (A_1 \cup A_2)) = \alpha_{w'}(G') + \delta_1$.

Therefore, $\alpha_w(G) = \delta_1 + \alpha_{w'}(G')$. Moreover, if S' is a MWSS of G, we may derive from it a MWSS of G, as soon as we are given: a MWSS of G^1; a MWSS of G^1 not intersecting A_1; a MWSS of G^1 not intersecting A_2; a MWSS of G^1 not intersecting $A_1 \cup A_2$ and a MWSS of G^1 not intersecting $A_1 \Delta A_2$ in case (ii). □

Trivially, we can apply the above procedure iteratively to each strip H_i. The problem of finding a MWSS on G reduces therefore to the same problem on the graph G^* that is the composition of $H'_1, \ldots, H'_k$ with respect to a suitable partition $\mathcal{P}^*$. The following lemma shows some key properties of G^*.

COROLLARY 2.9. *G^* is a line graph and in time $O(k)$ we can build a root graph $\tilde{G}$ with $O(k)$ vertices and edges.*

PROOF. It is trivial to see that the strips H'_i, $i = 1, ..., k$, are line strips, according to Definition 2.5, and therefore it follows from Lemma 2.7 that G^* is a line graph. (Note also that by construction G^* has at most $3k$ vertices). Moreover, the proof of the same lemma, together with Lemma 2.1 (which is constructive), suggests how to build a root graph for G^* with $O(k)$ vertices and edges in $O(k)$-time : we skip the details. □

Since the number k of strips is bounded by $O(|V(G)|)$, it follows that we have reduced, provided we can efficiently compute the weights w' for the vertices of each strip H'_i, the maximum weighted stable set problem on G to a weighted matching problem on the graph $\tilde{G}$, that has $O(|V(G)|)$ vertices and edges. This latter problem can be solved in time $O(|V(G)|^2 \log |V(G)|)$ by [Gabow 1990]. Also note that the computation of the weights w' for the vertices of some strip H'_i requires the solution of some MWSS problems on G^i, where, eventually, the weight of some vertex is set to 0. Thus, we have proved the following:

THEOREM 2.10. *[Oriolo et al. 2008]. The maximum weighted stable set problem on a graph G, that is the composition of some set of strips $(G^1, \mathcal{A}^1), \ldots, (G^k, \mathcal{A}^k)$, can be solved in $O(|V(G)|^2 \log |V(G)| + \sum_{i=1,\ldots,k} p_i(|V(G^i)|))$-time, if each G^i belongs to some class of graphs, where the same problem can be solved in time $O(p_i(|V(G^i)|))$.*

3. ARTICULATION CLIQUES

Most claw-free graphs admit a strip decomposition, and we will later show how to find such a decomposition. *Articulation cliques* are the main tool for this decomposition, as we will show that every claw-free graph G, that has some articulation clique, admits a strip decomposition where each partition-clique is indeed an articulation clique of G. We therefore devote this section to the study of articulation cliques. Their definition requires, however, a few results and definitions.

Let $G(V, E)$ be a claw-free graph. A vertex $v \in V$ such that $N[v]$ can be covered by two maximal cliques K_1 and K_2 (not necessarily different) is called *regular*, and it is called *strongly regular* when this covering is unique: in this case, we also say that K_1 and K_2 are *crucial* (*for* v). A vertex that is not regular is called *irregular*. Note that each irregular vertex of G is the center of an odd k-anti-wheel with $k \geq 5$, and that G is quasi-line if and only every vertex $v \in V$ is regular. The following classical lemma of Fouquet shows that 5-wheels are the only "obstruction" to regularity when $\alpha(G)$ is large enough.

LEMMA 3.1. *[Fouquet 1993] If G is a connected claw-free graph G with $\alpha(G) \geq 3$, then each vertex is either regular or is the center of a 5-wheel. Moreover, if $\alpha(G) \geq 4$, then G has no odd k-anti-wheel, with $k > 5$.*

The following lemma goes along the same lines of Corollary 4 in [Kennedy and King 2011]. It makes use of the following fact, observed by Kloks, Kratsch and Müller [2000].

FACT 3.2. *In a claw-free graph every vertex has at most $2\sqrt{|E|}$ neighbors. In particular every clique has size at most $\sqrt{|E|}$.*

LEMMA 3.3. *Let $G(V, E)$ be a claw-free graph.*

(i) For each vertex $v \in V$, in time $O(|E|)$ we may recognize if v is regular, strongly regular (and, in this case, find the cliques that are crucial for v and store them sorted with respect to any given linear order on V), or irregular (and, in this case, find an odd k-anti-wheel centered in v, $k \geq 5$).
(ii) In time $O(|V||E|)$ we may either recognize that G is quasi-line, or that $\alpha(G) \leq 3$, or build, for each irregular vertex a, a 5-wheel $W(a)$ centered in a.

PROOF. (i) For each $v \in V$, consider the graph $H = \overline{G[N(v) \setminus U(v)]}$. Because of Fact 3.2, H can be build in time $O(|E|)$ and has at most $|E|$ edges. Then v is regular if H is bipartite and is irregular otherwise. Bipartiteness can be checked in time $O(|E|)$ by adapting breadth first search. Also, it can be modified as to return an odd chordless cycle. If H is bipartite, then v is strongly regular if and only H is connected: in this case, $S_1 \cup U[v]$ and $S_2 \cup U[v]$ are the crucial cliques for v, where S_1 and S_2 are the classes of the unique bi-coloring of H. Once one has obtained S_1, S_2, if we are given a linear order on the set V, we can sort them in $O(|E|)$-time using Fact 3.2. If H is not bipartite, then the vertices of any odd chordless cycle of H together with v induce an odd k-anti-wheel on G, with $k \geq 5$, since H has no triangles. The statement trivially follows.
(ii) It follows from (i) that in time $O(|V||E|)$ we either recognize that G is quasi-line or build, for each irregular vertex a, an odd k-anti-wheel $W(a)$ centered in a, $k \geq 5$. If there exists an irregular vertex a such that $W(a)$ is a long odd anti-wheel, then $\alpha(G) \leq 3$ by Lemma 3.1. □

Following Lemma 3.3, in the rest of this section we assume that for each vertex, the crucial cliques are stored sorted with respect to some given ordering on V. The following technical lemmas give some sufficient conditions to recognize strongly regular vertices.

LEMMA 3.4. *Let G be a claw-free graph, K be a maximal clique of G and $v \in K$. If either v is simplicial, or $N(v) \setminus K$ is anti-complete to some vertex in K, then v is strongly regular and K is crucial for v.*

PROOF. The statement is trivial if v is simplicial. So suppose that $N(v) \setminus K$ is non-empty and there exists a vertex $w \in K$ such that $N(v) \setminus K$ is anti-complete to w. This implies that $N(v) \setminus K$ is a clique, otherwise any stable set of size two in $N(v) \setminus K$, say $\{t, z\}$, would cause the claw $(v; w, t, z)$. Thus, v is regular; we now show that v is strongly regular and K is crucial for v. In order to prove this, it is enough to show that, if K_1, K_2 are two maximal cliques covering $N[v]$, then either $K = K_1$ or $K = K_2$. So let K_1, K_2 ($K_1 \neq K_2$ since v is not simplicial) be two maximal cliques covering $N[v]$; trivially, the set $U[v]$ satisfies $U[v] = K_1 \cap K_2$. We may assume without loss of generality that $w \in K_1$ and $N(v) \setminus K \subseteq K_2 \setminus K_1$. Let $u \in K \setminus U[v]$: then, there exists $z \in N(v) \setminus K : uz \notin E$. As $N(v) \setminus K \subseteq K_2$, it follows that $u \in K_1$. Therefore, $K \setminus U[v] \subseteq K_1$ and thus $K \subseteq K_1$. By maximality, $K = K_1$. □

LEMMA 3.5. *Let G be a claw-free graph and v a vertex of G. If there exist non-empty vertex disjoint cliques X_1, X_2, Y_1, Y_2 such that $X_1 \cup X_2 \cup Y_1 \cup Y_2 = N(v) \setminus U(v)$, X_1 is complete to X_2, Y_1 is* *complete to Y_2, X_1 is anti-complete to Y_2, X_2 is anti-complete to Y_1 and X_1 is not complete to Y_1, then v is strongly regular and $X_1 \cup X_2 \cup U[v]$ and $Y_1 \cup Y_2 \cup U[v]$ are crucial for v.*

PROOF. By hypothesis, v is regular. Let Q_1 and Q_2 two maximal cliques covering $N[v]$. We can assume without loss of generality that $X_1 \subseteq Q_1$ and $Y_2 \subseteq Q_2$. There exists $y \in Y_1$ that is not complete to X_1; therefore $y \in Q_2$, and then $X_2 \subseteq Q_1$ and $Y_1 \subseteq Q_2$. So $X_1 \cup X_2 \subseteq Q_1$ and $Y_1 \cup Y_2 \subseteq Q_2$. But then, trivially, $Q_1 = X_1 \cup X_2 \cup U[v]$ and $Q_2 = Y_1 \cup Y_2 \cup U[v]$. □

Definition 3.6. A maximal clique K of a claw-free graph G is an *articulation clique* if, for each $v \in K$, K is crucial for v. We denote by $\mathcal{K}(G)$ the family of all articulation cliques of G.

Note that, by definition, each vertex of an articulation clique is strongly regular. However, the converse does not hold, i.e., it is not true that a maximal clique made of strongly regular vertices is an articulation clique. Indeed, consider vertex v complete to a path $\{a_1, a_2, a_3, a_4\}$ of length three. The clique induced by $\{v, a_2, a_3\}$ is not an articulation clique, though it is maximal and each vertex is strongly regular.

We observe a simple fact that will be extensively used in the following.

FACT 3.7. *Let G be a claw-free graph and let $K \in \mathcal{K}(G)$ be an articulation clique of G. For each $v \in K$, $N(v) \setminus K$ is a clique.*

LEMMA 3.8. *Let $G(V, E)$ be a claw-free graph. G has at most $2n$ articulation cliques and we can list them all in time $O(|V||E|)$.*

PROOF. It follows from Lemma 3.3 that we may build in time $O(|V||E|)$ the set of strongly regular vertices of G and, for each vertex $v \in V$, its crucial cliques (if any). Remember we can assume the list of vertices in each crucial clique are sorted. Note that G has at most $2n$ crucial cliques (i.e., cliques that are crucial for some strongly regular vertex), as every vertex is contained in at most two crucial cliques. Then, building upon Fact 3.2, we may list all articulation cliques of G in time $O(|V||E|)$ by simply checking for every crucial clique K that every vertex v in the clique (those are at most $\leq \sqrt{|E|}$ by Fact 3.2) has K as a crucial clique (comparing K with the two crucial cliques of v takes $O(\sqrt{|E|})$ since all cliques are sorted). □

In the following, we characterize two classes of maximal cliques that are articulation cliques: cliques of claw-free graphs with a simplicial vertex; net cliques of quasi-line graphs (cf. Definition 1.6). (There are articulation cliques that do not belong to these classes, though; however, a complete characterization of articulation cliques in claw-free/quasi-line graphs is not necessary for the following.)

LEMMA 3.9. *Let $G(V, E)$ be a claw-free graph and u a simplicial vertex of G. Then $N[u]$ is an articulation clique.*

PROOF. Let $K = N[u]$. From Lemma 3.4, each simplicial vertex of K is strongly regular and K is crucial for it. Consider now a vertex $v \in K$ that is not simplicial, i.e., such that $N(v) \setminus K \neq \emptyset$. Then $N(v) \setminus K$ is anti-complete to u and it follows again from Lemma 3.4 that v is strongly regular and K is crucial for it. The statement follows. □

LEMMA 3.10. *In a quasi-line graph every net clique is an articulation clique.*

PROOF. Let G be quasi-line, $\{v_1, v_2, v_3; s_1, s_2, s_3\}$ a net of G and K a maximal clique such that $\{v_1, v_2, v_3\} \subseteq K$. For $i \in [3]$, let K_i be the set of vertices from K that are adjacent to s_i ($K_i \neq \emptyset$ since $v_i \in K_i$), and $K_4 := K \setminus (K_1 \cup K_2 \cup K_3)$. Note that $\{K_1, K_2, K_3, K_4\}$ is a partition of K, since a vertex $v \in K$ that is adjacent to two vertices from s_1, s_2, s_3, say s_1 and s_2, implies the claw $(v; s_1, s_2, v_3)$. We now show that any vertex $v \in K$ is strongly regular and K is crucial for v.

First, suppose $v \in K_1$. Let $\{Q_1, Q_2\}$ be a pair of maximal cliques such that $N[v] = Q_1 \cup Q_2$ (such a pair exists, since the graph is quasi-line). Assume without loss of generality that $s_1 \in Q_1$, it follows that $K \setminus K_1 \subseteq Q_2$. We now show that every vertex $z \in N(v) \setminus K$ is not complete to $K \setminus K_1$. Suppose the contrary, i.e., there exists $z \in N(v) \setminus K$ that is complete to $K \setminus K_1$. Since K is maximal, there exists $w \in K_1$, $w \neq v$, such that $wz \notin E$. Since z is adjacent to v, it cannot be adjacent to both s_2 and s_3 (otherwise there would be the claw $(z; s_2, s_3, v)$). Assume without loss of generality z is not linked to s_3. Let z_3 be a vertex in K_3. Then $z_3z \in E$, as z is complete to $K \setminus K_1$. Now since $ws_3 \notin E$, $(z_3; s_3, w, z)$ is a claw, a contradiction. Therefore, every vertex in $z \in N(v) \setminus K$ is not complete to $K \setminus K_1$ and so it must belong to Q_1. It follows that $Q_1 = (N(v) \setminus K) \cup U[v]$ and $Q_2 = K$ form the unique pair of maximal cliques covering $N[v]$, thus v is strongly regular and K is crucial for v. The same argument holds for any vertex $v \in K_2$ or $v \in K_3$.

Now suppose that $v \in K_4$. If v is a simplicial vertex, then from Lemma 3.4 v is strongly regular and K is crucial for v. Hence suppose that there exists $w \notin K$ such that $wv \in E$. Observe that w

is adjacent to at most one vertex of $\{s_1, s_2, s_3\}$: if the contrary, assume without loss of generality $s_1, s_2 \in N(w)$, there would be the claw $(w; v, s_1, s_2)$. Hence there exists a stable set of size three in $\{w, s_1, s_2, s_3\}$ containing w, say $\{w, s_1, s_2\}$. Then $(v_1, v_2, v; s_1, s_2, w)$ is another net, and with respect to this net, we are back to the previous case. □

3.1. The maximum weighted stable set problem in {claw, net}-free graphs

In this section, we show that we can solve the maximum weighted stable set problem in a graph G that has no articulation cliques, and therefore is {claw, net}-free, in time $O(|V||E|)$. Note that the algorithm given by Pulleyblank and Shepherd in [1993] for solving the MWSS in distance claw-free graphs can be used to solve the above problem in time $O(|V(G)|^4)$ (a graph is *distance claw-free*, if, for each vertex $v \in V(G)$ and j, $\alpha(N_j(v)) \leq 2$). Also Brandstadt and Dragan [2003] show that a maximum *cardinality* stable set in a {claw, net}-free graph G can be found in time $O(|V(G)|^3)$. In the following, we essentially build upon these results to get an $O(|V||E|)$-time algorithm for the weighted case.

We start with a definition. Recall (cf. Definition 1.5) that a vertex v of a connected graph G is *distance simplicial* if, for every j, $\alpha(N_j(v)) \leq 1$.

Definition 3.11. A vertex v of a connected graph G is *almost distance-simplicial* if $\alpha(N_j(v)) \leq 1$ for every $j \geq 2$, and $\alpha(N(v) \cup N_2(v)) \leq 2$. A graph is *almost distance-simplicial* if there exists v that is distance simplicial.

The next lemma builds upon a construction and an algorithm from Pulleyblank and Shepherd [1993] for distance-claw-free graphs.

LEMMA 3.12. *Let $G(V, E)$ be a connected graph and z an almost distance-simplicial vertex of G. The maximum weighted stable set problem in G can be solved in time $O(|V|^2)$. If G is also claw-free, the complexity reduces to $O(|E|)$.*

PROOF. We are given a graph $G(V, E)$ and a weight function $w : V(G) \mapsto \mathbb{R}$. Let $p \in \mathbb{N}$ be the minimum number i such that $N_i(z) = \emptyset$. We write $N_0(z) = \{z\}$ and denote by $\mathcal{S}_i, i = 0, ..., p-1$ the family of all stable sets in $G[N_i(z)]$ (note that the empty set is included in each $\mathcal{S}_i$).

We now associate to G an auxiliary directed graph $D(V(D), A(D))$. The set $V(D)$ consists of $\{v^i_S : S \in \mathcal{S}_i, i = 0, ..., p\}$, together with two special nodes u^*, v^*. $A(D)$ is made of the following arcs: $(u^*, v^0_{\{z\}})$ and $(u^*, v^0_\emptyset)$; for each $i = 0, ..., p-2$ and S stable set of $G[N_i(z) \cup N_{i+1}(z)]$, the arc $(v^i_{S \cap N_i(z)}, v^{i+1}_{S \cap N_{i+1}(z)})$; for each $S \in \mathcal{S}_{p-1}$, the arc (v^{p-1}_S, v^*). We assign weights w' to the arcs of D as follows: for each arc $a = (x, v^*)$, $w'_a = 0$; for each other arc $a = (x, v^i_S)$, $w'_a = \sum_{y \in S} w_y$. The MWSS problem in G is equivalent to the longest directed (u^*, v^*)-path in the acyclic graph D and can thus be solved in time $O(|A(D)|)$, assuming that D is stored via adjacency lists (see e.g. [Ahuja et al. 1993]).

We now bound $|A(D)|$ and the time complexity to build the auxiliary graph. Let $n_i = |N_i(z)|$ for $i = 1, ..., p-1$. The auxiliary graph has $O((n_1)^2 + |V|)$ vertices. Note that $O((n_1)^2 + |V|)$ also suffices to *determine* those vertices. In order to build the auxiliary graph, we are left with computing its arcs; this requires: for $i = 1, ..., p-2$, checking all pairs of vertices (v^i_S, v^{i+1}_T) where S is a stable set of size one in $\mathcal{S}_i$ and T is a stable set of size one in $\mathcal{S}_{i+1}$ (and, in case, adding the corresponding arc); for $i = 0, ..., p-1$, adding for all $T \in \mathcal{S}_{i+1}$ the arc between $v^i_\emptyset$ and v^{i+1}_T and for all $S \in \mathcal{S}_i$, the arc between v^i_S and $v^{i+1}_\emptyset$; adding two arcs $(u^*, v^0_{\{z\}})$ and $(u^*, v^0_\emptyset)$; for each $S \in \mathcal{S}_{p-1}$, adding the arc (v^{p-1}_S, v^*). Hence building the graph and storing it via adjacency lists requires $O((n_1)^2 + |V| + n_1 \cdot n_2 + ... + n_{p-2} \cdot n_{p-1})$-time. But $O(n_1 \cdot n_2 + ... + n_{p-2} \cdot n_{p-1})$ is bounded by $O((\max\{n_1, n_2\})^2 + ... + (\max\{n_{p-2}, n_{p-1}\})^2) \leq O(2(n_1)^2 + ... + 2(n_{p-1})^2)$. Also, $n_1 \leq |V|$ ($n_1 \leq \sqrt{|E|}$ for claw-free graphs by Fact 3.2). Moreover, because for all $i \geq 2$, $N_i(z)$ is a clique of G with $O((n_i)^2)$ edges, we have $O(|E|) \geq O(\sum_{i \geq 2} (n_i)^2)$. It follows that the auxiliary graph can be built and stored via adjacency lists in time $O(|V|^2)$ ($O(|E|)$ for claw-free graphs) and $|A(D)| = O(|V|^2)$ ($|A(D)| = O(|E|)$ for claw-free graphs). The statement follows.

□

We need a few more results from the literature.

Definition 3.13. A triple $\{x, y, z\}$ of vertices of a graph G is an asteroidal triple (AT) if for every two of these vertices there is a path between them avoiding the closed neighborhood of the third. A graph G is called asteroidal triple-free (AT-free) if it has no asteroidal triple.

Brandstadt and Dragan [2003] proved the following:

LEMMA 3.14. *For every vertex v in a {claw,net}-free graph $G(V, E)$, $G[V \setminus N[v]]$ is {claw,AT}-free.*

Now using the celebrated 2LexBFS algorithm, Hempel and Krastch [2002] proved the additional following result (Lemma 6 in [Hemper and Kratsch 2002]):

LEMMA 3.15. *Given a {claw,AT}-free graph $G(V, E)$, one can find in $O(|E|)$ an almost distance simplicial vertex in G.*

THEOREM 3.16. *The maximum weighted stable set problem on a graph $G(V, E)$ that is {claw,net}-free can be solved in time $O(|V||E|)$.*

PROOF. For each $v \in V$ we compute the MWSS picking v by solving the MWSS on $G(V \setminus N[v])$. This can be done in $O(|E|)$ time, because of Lemma 3.14, Lemma 3.15 and Lemma 3.12. We choose the best stable set over the $|V|$ choices: the results follows. □

4. AN ALGORITHMIC DECOMPOSITION OF QUASI-LINE GRAPHS

In this section, we provide an algorithmic decomposition of *quasi-line graphs* (we will deal with *claw-free* graphs in the next section). We will show that most quasi-line graphs admit a strip decomposition where each partition-clique is an articulation clique of G. In order to find this decomposition, we therefore need to reverse the operation of composition and define a suitable operation of "ungluing" of articulation cliques.

Let K be an articulation clique of a *quasi-line* graph G. The ungluing of K requires a partition of the vertices of K into suitable classes. These classes are the equivalence classes defined by an equivalence relation $\mathcal{R}$ on the vertices of K. Call *bound* a vertex of K that belongs to two distinct articulation cliques of G (note that no vertex belongs to *more* than two articulation cliques). Then, for $u, v \in K$, $u\mathcal{R}v$ if and only if:

(i). either $u = v$;
(ii). or $u \neq v$, both u and v are bound and they belong to the same articulation cliques (note that in this case u and v are true twins);
(iii). or $u \neq v$, u and v are neither simplicial nor bound and $(N(v) \setminus K) \cup (N(u) \setminus K)$ is a clique.

We claim that $\mathcal{R}$ define an equivalence relation on the vertices of K. In fact, while symmetry and reflexivity of $\mathcal{R}$ are by definition, transitivity follows either from definition or from the next lemma.

LEMMA 4.1. *Let $G(V, E)$ be a quasi-line graph, K an articulation clique with three distinct non-simplicial vertices $u, v, z \in K$. If $(N(u) \setminus K) \cup (N(z) \setminus K)$ and $(N(v) \setminus K) \cup (N(z) \setminus K)$ are cliques, then also $(N(u) \setminus K) \cup (N(v) \setminus K)$ is a clique.*

PROOF. In the following, for $y \in K$, we let $\widetilde{N}(y) = N(y) \setminus K$. Since u, v, z are non simplicial, it follows that $\widetilde{N}(u)$, $\widetilde{N}(v)$ and $\widetilde{N}(z)$ are non-empty. Now suppose the statement is false; therefore there exist w_1 and $w_2 \in \widetilde{N}(u) \cup \widetilde{N}(v)$ that are non-adjacent. Since $\widetilde{N}(u)$ and $\widetilde{N}(v)$ are cliques, it follows that without loss of generality $w_1 \in \widetilde{N}(u) \setminus \widetilde{N}(v)$ and $w_2 \in \widetilde{N}(v) \setminus \widetilde{N}(u)$. Furthermore, note that $w_1, w_2 \notin \widetilde{N}(z)$, since this would contradict the hypothesis. Then pick any vertex t from $\widetilde{N}(z)$: $tz, tw_1, tw_2 \in E$, and $w_1z, w_2z \notin E$ hold; thus, $(t; z, w_1, w_2)$ is a claw, a contradiction. □

The above discussion shows that the following Definition 4.2 is consistent. We then skip the straightforward proof of Lemma 4.3.

Definition 4.2. Let G be quasi-line and K an articulation clique of G. We denote by $\mathcal{Q}(K)$ the family of the equivalence classes defined by $\mathcal{R}$ and call each class of $\mathcal{Q}(K)$ a *spike* of K.

LEMMA 4.3. *Let $G(V, E)$ be a quasi-line graph, K an articulation clique of G and Q a spike of K. Then:*

— *either $Q = \{v\}$, for some simplicial vertex v of G. In this case, $N[Q] = K$ and the spike is called* simplicial*;*
— *or $Q = U[v]$ for some bound vertex v of G. In this case, the vertices in Q are true twins, the unique pair of maximal clique covering $N[Q]$ is $\{K, (N(Q) \setminus K) \cup Q\}$, where $(N(Q) \setminus K) \cup Q$ is also an articulation clique of G, and the spike is called a* bound *spike;*
— *or Q is made of a subset of non-bound and non-simplicial vertices of K. In this case, for each $v \in Q$, the unique pair of maximal clique covering $N[v]$ is $\{K, (N(v) \setminus K) \cup U[v]\}$, and the spike is called* non-trivial*.*

Before proceeding further, it is convenient to shed some light on the intersections between spikes from different articulation cliques. We will denote by $\mathcal{Q}(\mathcal{K}(G))$ the disjoint union of all spikes of articulation cliques of G: note that $\mathcal{Q}(\mathcal{K}(G))$ is in general a multi-family. The following lemma is straightforward.

LEMMA 4.4. *Let Q_1 and Q_2 be spikes of different articulation cliques. Then either $Q_1 \cap Q_2 = \emptyset$, or Q_1 and Q_2 are bound spikes and $Q_1 = Q_2$.*

We are ready to introduce the operation of ungluing of articulation cliques in quasi-line graphs.

Definition 4.5. Let G be a quasi-line graph. The *ungluing* of the cliques in $\mathcal{K}(G)$ consists of removing, for each articulation clique $K \in \mathcal{K}(G)$, the edges between different spikes of $\mathcal{Q}(K)$. We denote the resulting graph by $G|_{\mathcal{K}(G)}$.

LEMMA 4.6. *Let $G(V, E)$ be a quasi-line graph. We can build the graph $G|_{\mathcal{K}(G)}$ and the family $\mathcal{Q}(\mathcal{K}(G))$ in time $O(|V||E|)$.*

PROOF.
Let $n = |V|$ and $m = |E|$. We can assume without loss of generality that G is connected, thus $m = \Omega(n)$. We know from Lemma 3.8 that $|\mathcal{K}(G)| \leq 2n$ and that we can list all cliques in $\mathcal{K}(G)$ in time $O(nm)$. For an articulation clique K and a vertex $u \in K$, we let $\widetilde{N}(u) := N(u) \setminus K$. Given a non empty set $S \subset V$, we denote by $\delta(S) := \{(u, v) \in E : u \in S, v \in V \setminus S\}$. We start by giving a bound on the time complexity needed to build $\mathcal{Q}(K)$ for a given articulation clique K.

So let K be an articulation clique of G. For each $u \in K$, we can build the set $\widetilde{N}(u)$, and check whether u is simplicial, in time $O(n)$. Note that $O(n)$ also suffices to check whether u is bound (in fact u is bound if and only if it belongs to another articulation clique, and $|\mathcal{K}(G)| \leq 2n$). Therefore, in time $O(|K|n)$ we can “pre-process” the clique, as to find its simplicial and bound vertices, as well the spikes defined by those vertices. We now compute the other spikes of K, i.e., the spikes with non-simplicial and non-bound vertices. In the following, we refer to the subset of non-simplicial and non-bound vertices of K as $\overline{K}$. We can also build $\widetilde{N}(\overline{K}) := \bigcup_{u \in \overline{K}} \widetilde{N}(u)$ in time $O(|K|n)$. By definition, two vertices $u, v \in \overline{K}$ belong to the same spike if and only if $\widetilde{N}(u) \cup \widetilde{N}(v)$ is a clique.

In order to compute the spikes for the vertices in $\overline{K}$, we construct a bipartite graph $G'(\overline{K} \cup \widetilde{N}(\overline{K}), E')$ such that $u \in \overline{K}$ and $w \in \widetilde{N}(\overline{K})$ are adjacent if and only if either $w \in \widetilde{N}(u)$, or w is complete to $\widetilde{N}(u)$. Note that each graph G' can be constructed by adding, for each $u \in \overline{K}$, the edges between u and all $w \in \widetilde{N}(u)$, and by checking for each $u \in \overline{K}$, $w \in \widetilde{N}(\overline{K})$ and $z \in \widetilde{N}(u)$ if $wz \in E$. Since the sets $\widetilde{N}(\overline{K})$ and $\widetilde{N}(u)$ for each $u \in \overline{K}$ are available, it follows that G' can be constructed in time $O(\sum_{u \in \overline{K}} \sum_{w \in \widetilde{N}(\overline{K})} \sum_{z \in \widetilde{N}(u)} 1) = O(\sum_{u \in \overline{K}} \sum_{z \in \widetilde{N}(u)} |\widetilde{N}(\overline{K})|) \leq O(n \cdot \sum_{u \in \overline{K}} \sum_{z \in \widetilde{N}(u)} 1) = O(n|\delta(K)|)$.

Hence, we can build $\mathcal{Q}(K)$ in time $O(n|K| + n|\delta(K)|)$ and consequently $\mathcal{Q}(\mathcal{K}(G))$ in $O(n^2 + nm)-$time, where we used the fact that $\sum_{K \in \mathcal{K}(G)} |K| = O(n)$ and $\sum_{K \in \mathcal{K}(G)} |\delta(K)| \leq 4m$ since any vertex is in at most 2 articulation cliques and thus any edge belongs to at most 4 sets from $\{\delta(K) : K \in \mathcal{K}(G)\}$. The statement follows. □

The following lemma, whose long proof is postponed to the appendix, shows several properties of the graph $G|_{\mathcal{K}(G)}$, that are crucial for the decomposition algorithm.

LEMMA 4.7. *Let G be a connected quasi-line graph with $\mathcal{K}(G) \neq \emptyset$ and let $\mathcal{C}$ be the set of connected component of $G_{|\mathcal{K}(G)}$. Then:*

(i) $G|_{\mathcal{K}(G)}$ is quasi-line.

(ii) If $Q \in \mathcal{Q}(\mathcal{K}(G))$, then Q entirely belongs to some connected component $C \in \mathcal{C}$, and C is distance simplicial with respect to Q. Moreover, if Q is either simplicial or bound, then $V(C) = V(Q)$.

(iii) If $Q_1, Q_2 \in \mathcal{Q}(\mathcal{K}(G))$ are non-trivial spikes belonging to a same connected component $C \in \mathcal{C}$, then there exists j_2 such that $Q_2 \subseteq N_{j_2-1}(Q_1) \cup N_{j_2}(Q_1)$ and $N_{j_2+1}(Q_1) = \emptyset$, where $N_j(Q_1)$ is the $j-$th neighborhood of Q_1 in C (and, analogously, there exists j_1 such that $Q_1 \subseteq N_{j_1-1}(Q_2) \cup N_{j_1}(Q_2)$ and $N_{j_1+1}(Q_2) = \emptyset$).

(iv) For each $C \in \mathcal{C}$, there are either one or two spikes from $\mathcal{Q}(\mathcal{K}(G))$ that belong to C. Therefore, if we let $\mathcal{A}(C)$ be the multi-family of these spikes, then $(C, \mathcal{A}(C))$ is a strip.

(v) G is the composition of the strips $\{(C, \mathcal{A}(C)) : C \in \mathcal{C}\}$ with respect to the partition $\mathcal{P}$ that puts two extremities in the same class if and only if they are spikes from a same articulation clique of G.

We are now ready to give our algorithmic decomposition theorem for quasi-line graphs.

THEOREM 4.8. *Let $G(V, E)$ be a connected quasi-line graph. In time $O(|V||E|)$ Algorithm 1:*

(j) either recognizes that G has no articulation cliques and therefore is net-free;

(jj) or provides a decomposition of G into $k \leq |V|$ quasi-line strips $(G^1, \mathcal{A}^1), \ldots, (G^k, \mathcal{A}^k)$, with respect to a partition $\mathcal{P}$ such that the partition-cliques are all articulation cliques of G. Moreover, for each strip $(G^i, \mathcal{A}^i)$, each extremity $A \in \mathcal{A}^i$ is a spike from some articulation clique of G and the graph G^i is distance simplicial with respect to A.

In particular, if $\mathcal{A}^i = \{A_1, A_2\}$, then:
— either $A_1 = A_2 = V(G^i)$;
— or $A_1 \cap A_2 = \emptyset$ and there exists j_2 such that $A_2 \subseteq N_{j_2-1}(A_1) \cup N_{j_2}(A_1)$ and $N_{j_2+1}(A_1) = \emptyset$, where $N_j(A_1)$ is the $j-$th neighborhood of A_1 in G^i (and, analogously, there exists j_1 such that $A_1 \subseteq N_{j_1-1}(A_2) \cup N_{j_1}(A_2)$ and $N_{j_1+1}(A_2) = \emptyset$).

ALGORITHM 1: Decomposition of quasi-line graphs

Require: A connected quasi-line graph G.
Ensure: The algorithm either recognizes that G has no articulation cliques and therefore is net-free, or returns a strip decomposition of G as to satisfy part (jj) of Theorem 4.8.

1: Find the family $\mathcal{K}(G)$ of all articulation cliques of G. If $\mathcal{K}$ is empty, then G has no net cliques and then it is net-free, **stop**.
2: Following Definition 4.5, unglue the articulation cliques in $\mathcal{K}(G)$ as to build the graph $G|_{\mathcal{K}(G)}$.
3: Let $\mathcal{C}$ be the components of $G|_{\mathcal{K}(G)}$. For each component $C \in \mathcal{C}$, let $\mathcal{A}(C)$ be the family (possibly a multi-family) of spikes in C.
4: **Return** the family of strips $\{(C, \mathcal{A}(C)) : C \in \mathcal{C}\}$ and the partition $\mathcal{P}$ of $\bigcup_{C \in \mathcal{C}} \mathcal{A}(C)$ that puts two extremities in the same class if and only if they are spikes from a same articulation clique.

Proof of Theorem 4.8. We first show that the algorithm is correct. If $\mathcal{K}(G) = \emptyset$, it follows from Lemma 3.10 that G is net-free. Otherwise, we rely on Lemma 4.7, and it is easy to check that (jj) holds, following statements $(i) - (v)$ in Lemma 4.7. In particular, as from (iv) each component of $G|_{\mathcal{K}(G)}$ defines a strip, there are at most $|V|$ strips. We now move to complexity issues. We can find all articulation cliques of G in time $O(|V||E|)$ thanks to Lemma 3.8. Moreover, we can build the graph $G|_{\mathcal{K}(G)}$ and the family $\mathcal{Q}(\mathcal{K}(G))$ in time $O(|V||E|)$, thanks to Lemma 4.6; this also immediately gives the partition $\mathcal{P}$. Finally, we can easily compute $\mathcal{C}$ in time $O(|E|)$ and the sets $\mathcal{A}(C)$ for each $C \in \mathcal{C}$ in time $O(|V|^2)$ (we have at most $|V|$ spikes and $|V|$ components and for each spike it is enough to check that one of its vertices is in a given component or not) .

See Figure 1 for the application of Algorithm 1 to an example.

4.1. The maximum weighted stable set problem in quasi-line graphs

We are almost ready to present an $O(|V|(|E| + |V| \log |V|))$-time algorithm for the weighted stable set problem on a quasi-line graph G. In fact, it follows from Theorem 4.8 and Theorem 2.10 that, in order to get an algorithm for the MWSS problem on a quasi-line graph G, we are left with showing how to find a MWSS in a graph H that is distance simplicial with respect to some clique. The next lemma shows that this can be done in time $O(|E(H)|)$. We skip the proof, as it goes along the same lines of that of Lemma 3.12.

LEMMA 4.9. *Let H be a connected graph and K a clique of H such that H is distance simplicial with respect to K. The maximum weighted stable set problem in H can be solved in time $O(|V(H)|^2)$. If G is also claw-free, the complexity reduces to $O(|E(H)|)$.*

The algorithm to solve the MWSS problem in a quasi-line graph is therefore the following:

ALGORITHM 2: Solution of the MWSS in quasi-line graphs

Require: A connected quasi-line graph $G(V, E)$ and a function $w : V \mapsto \mathbb{R}$.
Ensure: The algorithm finds a maximum weighted stable set in G with respect to w in $O(|V|(|E| + |V| \log |V|))$-time.

1: Use Algorithm 1 to either detect in $O(|V||E|)$-time that G is {claw,net}-free or to provide in the same time a decomposition of G that obeys Theorem 4.8.
2: If G is {claw,net}-free, then use Theorem 3.16 to solve the problem in $O(|V||E|)$ time.
3: Else, G is the composition of the k strips $H_1, ..., H_k$. For each strip, a MWSS can be found in time $O(|E(H_i)|)$, by Lemma 4.9. Then, by Theorem 2.10, a MWSS in G can be computed in time $O(|V|^2 \log |V|)$.

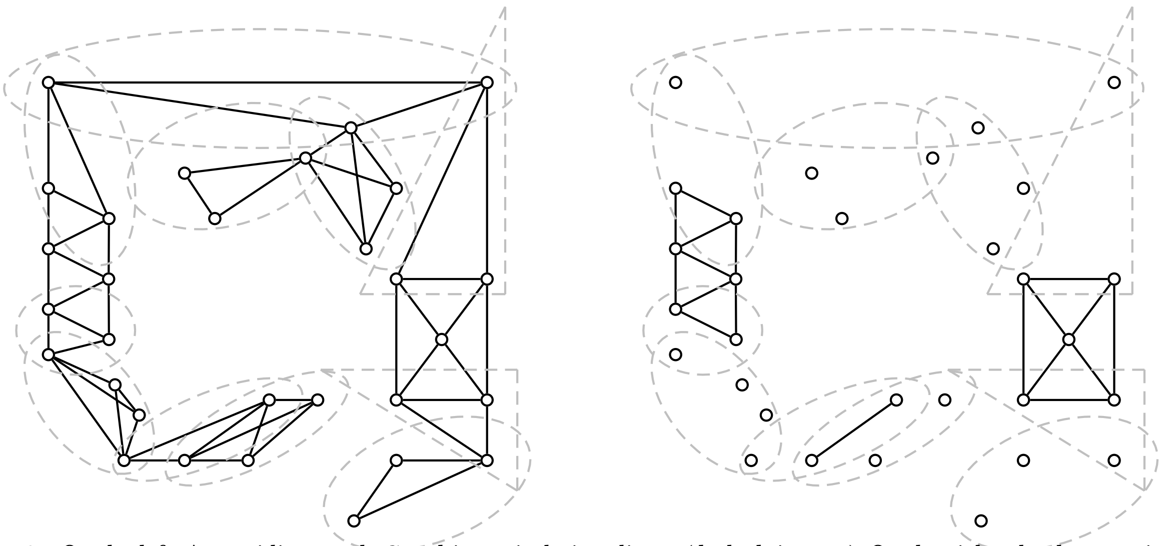

Fig. 1. On the left: A quasi-line graph G and its articulation cliques (dashed, in gray). On the right: the decomposition of G into distance simplicial strips obtained using Algorithm 1: the articulation cliques of G are now the partition-cliques of its decomposition. Recall that, for each strip $(C, \mathcal{A}(C))$, at most two different partition-cliques K_1, K_2 have non-empty intersection with $V(C)$. Thus, $K_1 \cap V(C)$ and $K_2 \cap V(C)$ (if K_2 exists) form the extremities from the set $\mathcal{A}(C)$.

5. AN ALGORITHMIC DECOMPOSITION OF CLAW-FREE GRAPHS

Let $G(V, E)$ be a claw-free graph. We know from Lemma 3.3 that in time $O(|V||E|)$ we may recognize that G is quasi-line (and, in this case, we can rely on Theorem 4.8 to get a finer decomposition), or that G is non-quasi-line and $\alpha(G) \leq 3$, or provide, for each irregular vertex $a \in V$, a 5-wheel $W(a)$ centered in a.

The main result of this section are Algorithm 3 and Theorem 5.6 showing how to deal with the latter case and produce a strip decomposition of G in strips that are distance simplicial and quasi-line (i.e., the same strips we have in Theorem 4.8) or non-quasi-line and with small stability number. However, the way we now build the strip decomposition is slightly different than for quasi-line graphs. In fact, Algorithm 1 produces the strips altogether (in a way, simultaneously) from the ungluing of the articulation cliques of G. Unfortunately, when dealing with claw-free graphs, the ungluing operation may create claws and so we have to take a (short) detour. Algorithm 3 will first iteratively find and suitably remove a family $\mathcal{H}$ of non-quasi-line strips of G, that we call *hyper-line* strips, as to produce a quasi-line graph $G|_{\mathcal{H}}$; then it will proceed as Algorithm 1 and build a strip decomposition $(\mathcal{F}, \mathcal{P})$ of $G|_{\mathcal{H}}$. Finally, it will suitably "combine" the strips in $\mathcal{F} \cup \mathcal{H}$ and the partition $\mathcal{P}$ to derive a strip decomposition of G. It turns out that the partition-cliques of this final decomposition are, in general, a *subset* of the articulation cliques of G, i.e., possibly there are some articulation cliques of G that we do not unglue.

We start with the crucial definition of hyper-line strips.

Definition 5.1. Let $G(V, E)$ be a claw-free graph and $(H, \mathcal{A})$ a strip that is either a 1-strip or a 2-strip with vertex disjoint extremities. We say that H is an hyper-line strip of G if:

— H is an induced subgraph of G, i.e., $H = G[V(H)]$;
— the core $C(H, \mathcal{A})$ of the strip $(H, \mathcal{A})$ is anti-complete to $V \setminus V(H)$;
— for each $A \in \mathcal{A}$, $A \cup (N(A) \setminus V(H))$ is an articulation clique of G
(NB : it is in particular a maximal clique).

Observe that, if $(H, \mathcal{A})$ is an hyper-line strip of G, then G is the composition of the strips $(H, \mathcal{A})$ and $(G \setminus V(H), \mathcal{K}(A))$, with respect to the partition $\{\{A, K(A)\}, A \in \mathcal{A}\}$. Here we let, for each $A \in \mathcal{A}$, $K(A) = N(A) \setminus V(H)$ and $\mathcal{K}(A) := \{K(A), A \in \mathcal{A}\}$.

Definition 5.2. Let $G(V, E)$ be a claw-free graph and let $(H, \mathcal{A})$ be an hyper-line strip of G. We denote by $G|_{(H,\mathcal{A})}$ the graph obtained from G by deleting the vertices in the core of the strip and, in case $(H, \mathcal{A})$ is a 2-strip, the edges between the extremities (if any), that is:

— $V(G|_{(H,\mathcal{A})}) = V(G) \setminus C(H,\mathcal{A})$;
— $E(G|_{(H,\mathcal{A})}) = \{uv \in E : u,v \in V(G|_{(H,\mathcal{A})}) \setminus \{uv : u \in A_1, v \in A_2, A_1 \neq A_2 \in \mathcal{A}\}$.

LEMMA 5.3. *Let G be a claw-free graph and $(H,\mathcal{A})$ an hyper-line strip of G. For each $A \in \mathcal{A}$, let $K(A) = N_G(A) \setminus V(H)$. Then:*

(i) $G|_{(H,\mathcal{A})}$ is claw-free.
(ii) Let v be a vertex of $G|_{(H,\mathcal{A})}$.
(iia) If $v \in A$, with $A \in \mathcal{A}$, then $N_{G|_{(H,\mathcal{A})}}[v] = A \cup K(A)$ and v is simplicial in $G|_{(H,\mathcal{A})}$;
(iib) else, $N_{G|_{(H,\mathcal{A})}}[v] = N_G[v]$. Moreover, $G|_{(H,\mathcal{A})}[N(v)] = G[N(v)]$, unless $\mathcal{A} = \{A_1, A_2\}$ and $v \in K(A_1) \cap K(A_2)$.
(iii) Let v be a vertex of $G|_{(H,\mathcal{A})}$. Then v is regular / strongly regular / irregular in $G|_{(H,\mathcal{A})}$ if and only if it is regular / strongly regular / irregular in G. In particular, if v is irregular and W is a 5-wheel of G centered in v, then W is also a 5-wheel of $G|_{(H,\mathcal{A})}$, and vice versa.
(iv) If K is an articulation clique of G that does not take vertices from $C(H,\mathcal{A})$, then K is also an articulation clique of $G|_{(H,\mathcal{A})}$, and vice versa.
(v) If $(\overline{H},\overline{\mathcal{A}})$ is an hyper-line strip of G that is vertex disjoint from $V(H)$, then $(\overline{H},\overline{\mathcal{A}})$ is also an hyper-line strip of $G|_{(H,\mathcal{A})}$ that is vertex disjoint from $V(H)$, and vice versa.

PROOF. (iia) and (iib) hold by construction and thus (ii) holds true. In particular, for any $v \in G|_{(H,\mathcal{A})}$, $N_{G|_{(H,\mathcal{A})}}[v] = N_G[v]$ unless $v \in A$ for $A \in \mathcal{A}$ or $\mathcal{A} = \{A_1, A_2\}$ and $v \in K(A_1) \cap K(A_2)$. Hence, vertices not in those two configurations stay regular / strongly regular or irregular if they previously were. Now, vertices of A for all $A \in \mathcal{A}$ are simplicial in $G|_{(H,\mathcal{A})}$ and thus strongly regular in $G|_{(H,\mathcal{A})}$ (as in G). Hence the only obstruction to (iii) can come from a vertex $v \in K(A_1) \cap K(A_2)$ when $\mathcal{A} = \{A_1, A_2\}$. But in this case, v is a strongly regular vertex of G, as it belongs to the articulation cliques $A_1 \cup K(A_1)$ and $A_2 \cup K(A_2)$. Since A_1, A_2 are vertex disjoint, removing adjacencies between A_1 and A_2 preserves regularity (the two cliques remains cliques) and strong regularity (if the partition was unique before, it remains unique). This proves (iii) and also proves that the graph remains claw-free, i.e., (i).

Consider now an articulation clique K of G that does not take vertices from $C(H,\mathcal{A})$. There are two cases: either K takes some vertex v from an extremity of $\mathcal{A}$, or it does not. In the former case, the only possibility is that $K == A \cup K(A)$, with $A \in \mathcal{A}$ (since $A \cup K(A)$ is an articulation clique of G). But $A \cup K(A)$ is then an articulation clique of $G|_{(H,\mathcal{A})}$ by Lemma 3.9 as v is simplicial in $G|_{(H,\mathcal{A})}$. Suppose now that K does not take any vertex from an extremity of $\mathcal{A}$. Observe that in this case no vertex v of K belongs to $K(A_1) \cap K(A_2)$, as otherwise v would belong to three articulation cliques of G. But then it follows from (iib) that $N_{G|_{(H,\mathcal{A})}}[v] = N_G[v]$ and $G|_{(H,\mathcal{A})}[N(v)] = G[N(v)]$, for each $v \in K$. Therefore, K is also an articulation cliques of $G|_{(H,\mathcal{A})}$. Reversing the previous arguments is straightforward, therefore statement (iv) holds.

Finally, let $(\overline{H},\overline{\mathcal{A}})$ be an hyper-line strip of G that is vertex disjoint from $V(H)$. Observe that $V(\overline{H}) \subset V(G|_{(H,\mathcal{A})})$ and, by statement (iib), $\overline{H}$ is an induced subgraph of $G|_{(H,\mathcal{A})}$ and the core of $(\overline{H},\overline{\mathcal{A}})$ is anti-complete to $V(G|_{(H,\mathcal{A})}) \setminus V(\overline{H})$. For each $\overline{A} \in \mathcal{A}$, let $K(\overline{A}) = N_G(\overline{A}) \setminus V(\overline{H})$. Observe that $K(\overline{A}) = N_{G|_{(H,\mathcal{A})}}(\overline{A}) \setminus V(\overline{H})$; it follows from statement (iv) that $\overline{A} \cup (N_{G|_{(H,\mathcal{A})}}(\overline{A}) \setminus V(\overline{H}))$ is an articulation clique of $G|_{(H,\mathcal{A})}$. Altogether, this shows that $(\overline{H},\overline{\mathcal{A}})$ is an hyper-line strip of $G|_{(H,\mathcal{A})}$. Reversing the previous arguments is straightforward, therefore statement (v) holds. □

As we discussed at the beginning of this section, we deal with a claw-free but not quasi-line graph $G(V,E)$ for which we are given, for each irregular vertex $a \in V$, a 5-wheel $W(a)$ centered in a. In order to get a strip decomposition of G we will iteratively find and remove (according to Definition 5.2) hyper-line strips as to end up with a quasi-line graph. The following crucial lemma, whose proof is given in the Appendix, shows that in a claw-free graph with simplicial vertices it is always possible to build an hyper-line strip "around" a 5-wheel.

LEMMA 5.4. *Let $G(V,E)$ be a connected claw-free graph and let $a \in V$ be the center of a 5-wheel $W(a)$ of G. Then:*

(i) either G has no simplicial vertices and $\alpha(G) \leq 3$;
(ii) or there exists an hyper-line strip $(H,\mathcal{A})$ such that $a \in C(H,\mathcal{A})$, no vertex of H is a simplicial vertex of G and $\alpha(H) \leq 3$.

Moreover, if we are given the 5-wheel $W(a)$ and the set of simplicial vertices of G, we can decide that (i) holds or find the hyper-line strip $(H,\mathcal{A})$ in time $O(|E|)$.

We now combine Lemma 5.3 and Lemma 5.4.

THEOREM 5.5. *Let $G(V, E)$ be a connected claw-free but not quasi-line graph and suppose that we are given, for each irregular vertex a, a 5-wheel $W(a)$ centered in a. In time $O(|V||E|)$ we may:*

(i) either recognize that G has no simplicial vertices and $\alpha(G) \leq 3$;
(ii) or build a family $\mathcal{H} = \{(H^i, \mathcal{A}^i), \}_{i=1}^t$ of vertex disjoint hyper-line strips of G such that:
— *each strip in $\mathcal{H}$ contains a 5-wheel of G, no simplicial vertices of G and has stability number at most 3;*
— *the graph $G|_\mathcal{H}$ obtained after iteratively reducing the strips in $\mathcal{H}$, i.e., $G|_\mathcal{H} := G^t$ where for $i \in [t]$ $G^i = G^{i-1}|_{(H^i,\mathcal{A}^i)}$ with $G^0 := G$ is quasi-line and $V(G|_\mathcal{H}) \neq \emptyset$.*

PROOF. Let a be an irregular vertex. From Lemma 5.4 we may either recognize that G has no simplicial vertex and $\alpha(G) \leq 3$, or find an hyper-line strip $(H^1, \mathcal{A}^1)$ such that $a \in C(H^1, \mathcal{A}^1)$ (thus $W(a) \subseteq V(H^1)$), no vertex of H^1 is simplicial and $\alpha(H^1) \leq 3$. If the former holds we are done, thus suppose it does not.

So, let G^1 be the graph $G|_{(H^1,\mathcal{A}^1)}$. It follows from statement (ii) of Lemma 5.3 that $Simp(G^1) = Simp(G) \cup \{v \in A, A \in \mathcal{A}^1\}$. Note that G^1 is not necessarily connected. However, if it is not, by construction, each component picks some vertex from an extremity of $(H^1, \mathcal{A}^1)$, and therefore each component has a simplicial vertex. Let C be a component of $G|_{(H^1,\mathcal{A}^1)}$. By (i) of Lemma 5.3, C is claw-free; by (iii) of the same lemma, either C is quasi-line, or we have, for each irregular vertex a of C, a 5-wheel $W(a)$ centered in a. Suppose that C is not quasi-line, and pick an irregular vertex a. As C contains some simplicial vertex, it follows from Lemma 5.4 that in C there exists an hyper-line strip $(H^2, \mathcal{A}^2)$ such that $a \in C(H^2, \mathcal{A}^2)$, no vertex of H^2 is a simplicial vertex of C and $\alpha(H^2) \leq 3$. Note in particular, that $(H^2, \mathcal{A}^2)$ and $(H^1, \mathcal{A}^1)$ are vertex disjoint: this is because the vertices of the core of $(H^1, \mathcal{A}^1)$ do not belong to $G|_{(H^1,\mathcal{A}^1)}$, while the vertices of the extremities of $(H^1, \mathcal{A}^1)$ are simplicial in $G|_{(H^1,\mathcal{A}^1)}$. Also, it follows from (v) of Lemma 5.3 that $(H^2, \mathcal{A}^2)$ is an hyper-line strip of G.

Then we proceed iteratively until does not exist irregular vertices, i.e., we define a series $G^1, \ldots, G^t$ of graphs such that, for $i \in [t]$, $G^i = G^{i-1}|_{(H^i,\mathcal{A}^i)}$ (we let $G^0 := G$), where, for $i \in [t]$:

— each graph G^i is claw-free and $Simp(G^i) = Simp(G) \bigcup_{j=1..i}\{v \in A, A \in \mathcal{A}^j\}$;
— $(H^i, \mathcal{A}^i)$ is an hyper-line strip of G^{i-1}, that is vertex disjoint from $(H^1, \mathcal{A}^1), \ldots, (H^{i-1}, \mathcal{A}^{i-1})$;
— H^i contains a 5-wheel of G^{i-1}, has stability number at most 3 and no simplicial vertices of G^{i-1};
— G^t is quasi-line.

(We remark that the first property easily follows from (ii) of Lemma 5.3.) By repeatedly applying Lemma 5.3, it follows that for each $i \in [t]$, $\{(H^i, \mathcal{A}^i)\}$ is an hyper-line strip of G, H^i contains a 5-wheel of G but no vertex from $Simp(G)$. Moreover, the graph G^t is non-empty, since the removal of each strip produces some simplicial vertex that will remain simplicial, so none of them belongs to any hyper-line strip that is removed, and consequently they belong to G^t. In order to conclude the proof, we must show that the family $\{(H^i, \mathcal{A}^i)\}_{i=1}^t$ can be found in time $O(|V||E|)$. Trivially, $t \leq |V|$, since the strips are vertex disjoint. Moreover, we may build the set of simplicial vertices of G in time $O(|V||E|)$, since $|N(v)| \leq 2\sqrt{|E|}$ for each $v \in V$. Finally, it follows from the hypothesis and from part (iii) of Lemma 5.3 that for each irregular vertex in G^i we are given a $5-$wheel centered in that vertex. Thus, Lemma 5.4 guarantees that we can build the family $\mathcal{H}$ in $O(|V||E|)$-time. □

So suppose that G is claw-free and that by Lemma 5.6 we have found a family $\mathcal{H}$ of vertex disjoint hyper-line strips of G such that $G|_\mathcal{H}$ is quasi-line. Following Algorithm 1, we decompose $G|_\mathcal{H}$ and get a strip decomposition $(\mathcal{F}, \mathcal{P})$, with $\mathcal{F} = \{(F^1, \mathcal{B}^1), \ldots, (F^k, \mathcal{B}^k)\}$, such that the partition-cliques are all articulation cliques of $G|_\mathcal{H}$ and, for each $i \in [k]$, the graph F^i is distance simplicial with respect to each $B \in \mathcal{B}^i$. We now show how to "combine" the strips in $\mathcal{F}$ with the strips in $\mathcal{H}$ as to derive a strip decomposition of G.

Let A be an extremity of a strip $(H, \mathcal{A}) \in \mathcal{H}$ and, as usual, let $K(A) = N_G(A) \setminus V(H)$. It follows from Theorem 5.5 that $A \cup K(A)$ is an articulation clique of $G|_\mathcal{H}$ and that the vertices of A are simplicial in $G|_\mathcal{H}$. Therefore, each vertex $v \in A$ determines a spike of $A \cup K(A)$ and, trivially, a 1-strip $(\{v\}, \{\{v\}\})$ of $\mathcal{F}$. Moreover, from Theorem 4.8 we know that the partition $\mathcal{P}$ is such that there is a class $P(A) \in \mathcal{P}$ corresponding to the partition-clique $A \cup K(A)$, that contains the extremities of the 1-strips $(\{v\}, \{\{v\}\}), v \in A$.

In order to get a strip decomposition for G from the strip decomposition $(\mathcal{F}, \mathcal{P})$ of $G|_\mathcal{H}$, it is therefore enough to do the following. For each strip $(H, \mathcal{A}) \in \mathcal{H}$:

— remove from $\mathcal{F}$ all 1-strips of the form $(\{v\}, \{\{v\}\})$, with $v \in A$ and $A \in \mathcal{A}$, and "replace" them with the strip $(H, \mathcal{A})$; this defines a new set of strips $\mathcal{F}'$.

— for each $A \in \mathcal{A}$, remove from the class $P(A) \in \mathcal{P}$ the extremities $\{\{v\}, v \in A\}$, and "replace" all of them with the extremity A; this defines a partition $\mathcal{P}'$ of the extremities of all strips in $\mathcal{F}'$.

Indeed every edge of G will appear either in a partition-clique from $\mathcal{P}'$ or in a strip of $\mathcal{F}'$ and vice versa, every edge of the strips in $\mathcal{F}'$ and edges of the partition-cliques from $\mathcal{P}'$ are edges of G.

We formalize the procedure outlined above in the following algorithm and theorem, that are the main results of this section. Once again, we recall that, given a claw-free graph $G(V, E)$, it follows from Lemma 3.3 that in time $O(|V||E|)$ we may recognize that G is quasi-line (and, in this case, we can rely on Theorem 4.8 to get a decomposition), or that G is non-quasi-line and $\alpha(G) \leq 3$, or provide, for each irregular vertex $a \in V$, a 5-wheel $W(a)$ centered in a.

ALGORITHM 3: Decomposition of a claw-free but not quasi-line graph

Require: A connected claw-free but not quasi-line graph G and, for each irregular vertex $a \in V$, a 5-wheel $W(a)$ centered in a.

Ensure: The algorithm either recognizes that G has stability number at most 3, or returns a strip decomposition of G as to satisfy statement (ii) of Theorem 5.6.

1: By Theorem 5.5, either conclude that G has stability number at most 3: **stop**, or build a family $\mathcal{H}$ of vertex disjoint hyper-line strips of G such that $G|_{\mathcal{H}}$ is quasi-line, and each strip has stability number at most 3, contains a 5-wheel of G but no simplicial vertices of G.
2: Use Algorithm 1 to find a strip decomposition $(\mathcal{F}, \mathcal{P})$ of $G|_{\mathcal{H}}$. Let $\mathcal{F}' = \mathcal{F}$ and $\mathcal{P}' = \mathcal{P}$.
3: For each A being an extremity of a strip $(H, \mathcal{A}) \in \mathcal{H}$ do:
remove from $\mathcal{F}'$ all 1-strips made of vertices from A, i.e., $\mathcal{F}' = \mathcal{F}' \setminus \{(\{v\}, \{\{v\}\}),\ v \in A\}$;
update the class $\mathcal{P}'$ by replacing the class $P(A)$ containing the extremities $\{\{v\}, v \in A\}$ with the class $(P(A) \cup A) \setminus \{\{v\}, v \in A\}$.
4: **Return** the family of strips $\mathcal{F}' \cup \mathcal{H}$ and the partition $\mathcal{P}'$.

THEOREM 5.6. *Let $G(V, E)$ be a connected claw-free but not quasi-line graphs and suppose that we are given, for each irregular vertex a, a 5-wheel centered in a. In time $O(|V||E|)$ Algorithm 3:*

(i) either recognizes that G has no simplicial vertices and $\alpha(G) \leq 3$;
(ii) or provides a decomposition of G into $k + t \leq |V|$ strips $(F^1, \mathcal{B}^1), \ldots, (F^k, \mathcal{B}^k), (H^1, \mathcal{A}^1), \ldots, (H^t, \mathcal{A}^t)$, with respect to a partition $\mathcal{P}$, such that each partition-clique is an articulation clique of G. Moreover:
  — *each graph H^j is a claw-free graph with an induced 5-wheel, no simplicial vertices of G and stability number at most 3;*
  — *each graph F^i is a quasi-line graph that is distance simplicial with respect to each $B \in \mathcal{B}^i$.*
    In particular, if $\mathcal{B}^i = \{B_1, B_2\}$, then:
    • *either $B_1 = B_2 = V(F^i)$;*
    • *or $B_1 \cap B_2 = \emptyset$ and there exists j_2 such that $B_2 \subseteq N_{j_2-1}(B_1) \cup N_{j_2}(B_1)$ and $N_{j_2+1}(B_1) = \emptyset$, where $N_j(B_1)$ is the j−th neighborhood of B_1 in F^i (and, analogously, there exists j_1 such that $B_1 \subseteq N_{j_1-1}(B_2) \cup N_{j_1}(B_2)$ and $N_{j_1+1}(B_2) = \emptyset$).*

Proof of Theorem 5.6. Correctness of Algorithm 3 easily follows from the above discussion. Note that the algorithm runs in $O(|V||E|)$-time, as its crucial steps 1 and 2 can be performed in $O(|V||E|)$-time. Lemma 5.3 proves that each partition-clique of G, which is an articulation clique of $G|_{\mathcal{H}}$, is an articulation clique of G. The statement then immediately follows, as soon as we use Theorem 4.8 for characterizing the quasi-line strips of $\mathcal{F}'$ and Theorem 5.5 for characterizing the strips containing a 5-wheel.

See Figure 2 for the application of Algorithm 3 to an example.

5.1. The maximum weighted stable set problem in claw-free graphs

Before giving our algorithm we need first the following simple result.

LEMMA 5.7. *We can enumerate all stable sets of a claw-free graph $G(V, E)$ with $\alpha(G) \leq 3$ in time $O(|V||E|)$.*

PROOF. Let S be a stable set of maximum cardinality. By maximality, S is a dominating set for G, i.e., $S \cup N(S) = V$. But by Fact 3.2, it follows that that $|V| \leq \alpha(G) + \alpha(G)\sqrt{|E|} \leq 3 + 3\sqrt{|E|}$, since $\alpha(G) \leq 3$. Thus one can enumerate all stable sets of G in time $O(|V|^3) = O(|V||E|)$. □

We are now ready to put all the bricks together and present our $O(|V|(|E| + |V| \log |V|))$-time algorithm for the maximum weighted stable set problem in a claw-free graph G.

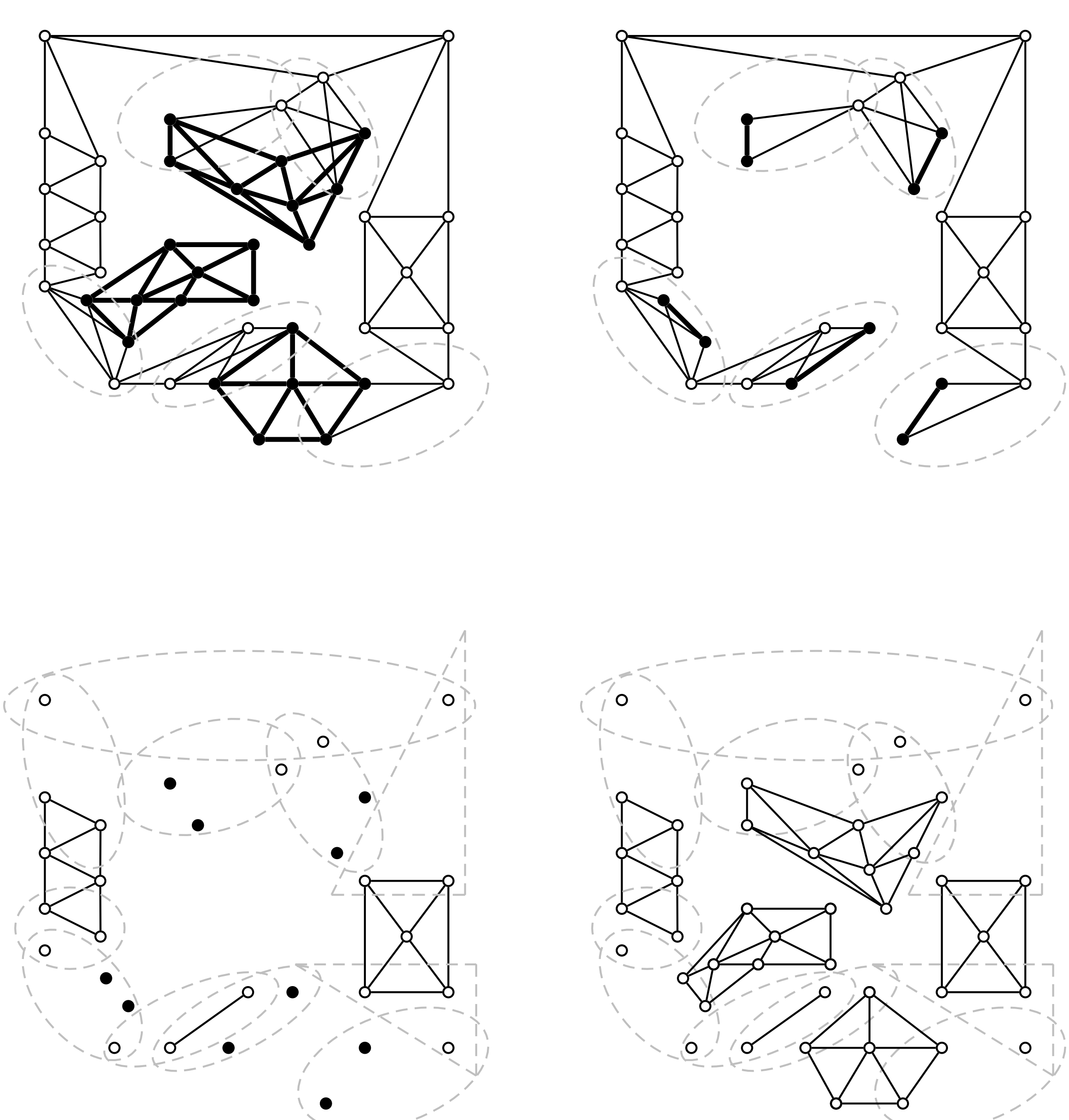

Fig. 2. On the top left: A claw-free graph G with three hyper-line strips (bold-faced) and the corresponding articulation cliques $A \cup K(A)$ for $A \in \mathcal{A}$ (dashed, in gray). On the top right: The quasi-line graph G' obtained from G after deleting the vertices in the core of each of those hyper-line strips (see Definition 5.2; vertices and edges from the hyper-line strips are still bold-faced; articulation cliques $A \cup K(A)$ for $A \in \mathcal{A}$ are dashed, in gray). Note that G' is the same graph of Figure 1. On the bottom left: The decomposition of the quasi-line graph G' into distance simplicial strips obtained using Algorithm 1: partition-cliques are dashed, in gray, and vertices from the hyperline strips are bold-faced. On the bottom right: The decomposition of the original claw-free graph G obtained using Algorithm 3: partition-cliques are dashed, in gray. Extremities of each strip can be identified as in Figure 1.

ALGORITHM 4: Solution of the MWSS in claw-free graphs

Require: A connected claw-free graph $G(V, E)$ and a function $w : V \mapsto \mathbb{R}$.
Ensure: The algorithm finds a maximum weighted stable set in G with respect to w in $O(|V|(|E| + |V| \log |V|))$-time.

1: Use Lemma 3.3 to recognize in $O(|V||E|)$-time that G is quasi-line, or to recognize that $\alpha(G) \leq 3$, or to find, for each irregular vertex $a \in V$, a 5-wheel $W(a)$ centered in a.
2: If in Step 1 we concluded that $\alpha(G) \leq 3$, find a MWSS by enumeration in $O(|V||E|)$-time, see Lemma 5.7: **stop**.
3: If G is quasi-line, use Algorithm 2 to find a MWSS in $O(|V|(|E| + |V| \log |V|))$-time: **stop**.
4: By Algorithm 3, in $O(|V||E|)$-time either recognize that $\alpha(G) \leq 3$ or provide a strip decomposition of G that obeys Theorem 5.6.
5: If in Step 4 we concluded that $\alpha(G) \leq 3$, find a MWSS by enumeration in $O(|V||E|)$-time, see Lemma 5.7: **stop**.
6: G is then the composition of (vertex disjoint) distance simplicial strips and strips with stability number less or equal to 3. In a distance simplicial strip $(F, \mathcal{B})$ a MWSS can be found in $O(|E(F)|)$-time by Lemma 4.9; in a strip $(H, \mathcal{A})$ such that $\alpha(H) \leq 3$ a MWSS can be found in $O(|V(H)||E(H)|)$-time by enumeration. Then a MWSS of G can be found in $O(|V|(|E| + |V| \log |V|))$-time from Theorem 2.10.

6. FINAL REMARKS AND OPEN QUESTIONS

We close the paper by summarizing some open problems stemming from our results.

To our view, the algorithms we presented show the potential of applying decomposition results (often available in structural graph theory) to the solution of combinatorial optimization problems. In particular, it seems that strip decompositions are very useful when dealing with claw free graphs; it is therefore an interesting question whether claw-free graphs is the only class of graphs where this kind of decomposition is useful. Analogously, it is an interesting question whether our algorithmic decomposition could be useful to solve problems other than the maximum weighted stable set.

We discussed in Section 1.1 that our decomposition theorem for claw-free graphs (Theorem 5.6) provide much less information about the structure of claw-free graphs than the results by Chudnovsky and Seymour (summarized in Theorem 1.4, see [Chudnovsky and Seymour 2008] for more details). On the other hand, for quasi-line graphs, the characterization given by Theorem 4.8 seems to be quite close to that provided by Theorem 1.2. While Theorem 4.8 has a direct and reasonably simple proof, Theorem 1.2 relies on the more general structure of claw-free graphs. We ask therefore whether (possibly, a sharpening of) Theorem 4.8 provides the same characterization from Theorem 1.2. (This would also give a direct proof of the latter characterization, thus answering a question already raised by King [2009].) In particular: which is the relation between quasi-line {claw,net}-free and fuzzy circular interval graphs ?

Last, recall that the weighted matching problem in a graph $H(W, F)$ can be solved in $O(|W|(|W| \log |W| + |F|)-$time [Gabow 1990], and we can consequently find a MWSS in a line graph $G(V, E)$ in time $O(|V|^2 \log |V|)$, while Algorithm 4 solves the MWSS in claw-free graphs in time $O(|V|(|V| \log |V| + |E|))$, i.e., slightly worse than for line graphs. Can we close this gap? We believe that this should be doable, in particular for quasi-line graphs. Also note that, except for the matching subroutine that relies on standard algorithms from the literature, Algorithm 4 only uses elementary data structures, so one could try to lower its complexity by switching to more sophisticated ones.

Appendix

A. THE PROOF OF LEMMA 4.7

This section is devoted to the proof of Lemma 4.7. Before going to the proof of the lemma itself, we prove an intermediate structural result.

LEMMA A.1. *Let $G(V, E)$ be a connected quasi-line net-free graph and K be a non-empty clique of G such that $N(K)$ is a clique, but $K \cup N(K)$ is not a clique. Then G is* distance simplicial *with respect to K.*

PROOF. Suppose by contradiction that there exists $j \geq 2$ such that $\alpha(N_j(K)) \geq 2$: we choose j to be minimal, i.e., for all $h < j$, $\alpha(N_h(K)) = 1$. Let $\{s_1, s_2\}$ be a stable set of size 2 in $N_j(K)$. For $i = 1, 2$, define the non-empty sets $S_i = N(s_i) \cap N_{j-1}(K)$, and note that $S_1 \cap S_2 = \emptyset$. In fact, suppose to the contrary there exists $v \in S_1 \cap S_2$; then, $(v; u, s_1, s_2)$ is a claw, for each u in $N(v) \cap N_{j-2}(K)$. This implies that (S, S_1, S_2) is a partition of $N_{j-1}(K)$, where we defined $S = N_{j-1}(K) \setminus (S_1 \cup S_2)$.

CLAIM 1. *For $i = 1, 2$, if $v \in S_i$ and $u \in N_{j-1}(K) \setminus S_i$, then $N(v) \cap N_{j-2}(K) \subseteq N(u) \cap N_{j-2}(K)$.*

PROOF. Suppose there exists a vertex w in $N_{j-2}(K)$ adjacent to v but not u, then $(v; w, u, s_i)$ is a claw, a contradiction. ■ □

CLAIM 2. $\cup_{i=1,2}(S_i \cup (N(S_i) \cap N_{j-2}(K)))$ *is a clique.*

PROOF. As $N_{j-1}(K)$ and $N_{j-2}(K)$ are cliques by hypothesis, it suffices to show that for any pair $u, v \in S_1 \cup S_2$, $N(u) \cap N_{j-2}(K) = N(v) \cap N_{j-2}(K)$. This immediately follows from Claim 1 for $u \in S_1$, $v \in S_2$. On the other hand, if $u, v \in S_1$, we pick $x \in S_2$ and then we have $N(u) \cap N_{j-2}(K) = N(x) \cap N_{j-2}(K) = N(v) \cap N_{j-2}(K)$. ■ □

CLAIM 3. $j = 2$.

PROOF. By hypothesis $j \geq 2$, as $N(K)$ is a clique. Now suppose $j \geq 3$, and let $v \in S_1$, $u \in S_2$, $w \in N(v) \cap N_{j-2}(K)$ and $s_3 \in N(w) \cap N_{j-3}(K)$. We already argued that $vs_2, us_1 \notin E$: moreover, by Claim 2, $w \in N(u)$ and by construction s_3 is non-adjacent to u, v, s_1, s_2, while w is non-adjacent to s_1, s_2. Thus, $\{v, u, w; s_1, s_2, s_3\}$ is a net, contradicting the hypothesis. ■ □

We conclude the proof by showing that, if $j = 2$, all vertices from K are simplicial, and thereto $K \cup N(K)$ is a clique, contradicting the hypothesis. Pick $u, v \in K$, and suppose there exists $w \in N[u] \setminus N[v]$. Recall that $N(K) = S_1 \cup S_2 \cup S$. Suppose first that $w \in S_1 \cup S_2$. Then, by Claim 2, v is anti-complete to $S_1 \cup S_2$, while u is complete to $S_1 \cup S_2$. This implies that $(u, v_1, v_2; s_1, s_2, v)$ is a net, for some vertices $v_1 \in S_1$ and $v_2 \in S_2$, i.e., a contradiction. Now let $w \in S$. Then, by Claim 1, v is anti-complete to $S_1 \cup S_2$. Recall that, by Claim 2, u is either complete or anti-complete to $S_1 \cup S_2$. If the former holds, then we can construct a net as done for the previous case. If conversely the latter holds, $(w, v_1, v_2; u, s_1, s_2)$ is a net. In both cases, we derive a contradiction. This shows that $N[u] = N[v]$ for arbitrary $u, v \in K$. As $N(K)$ is a clique, we conclude that $K \cup N(K)$ is a clique.

We now move to the proof of Lemma 4.7.

PROOF. In the following, for an articulation clique K of G, a vertex $v \in K$, and a set $Q \in \mathcal{Q}(K)$, we use the notation $\widetilde{N}_K(v) := N(v) \setminus K$ and $\widetilde{N}_K(Q) := N(Q) \setminus K$. We omit the subscript K when it is clear from the context. Note that with $N()$, $\mathcal{Q}()$,... we denote those sets in the graph G, while we add the subscript $G|_{\mathcal{K}(G)}$ when we refer to the corresponding sets in the graph $G|_{\mathcal{K}(G)}$. We start with some basic facts on articulation cliques.

CLAIM 4. *Let $K \in \mathcal{K}(G)$ be an articulation clique.*

(j) If Q_1 is a non-trivial spike of K and Q_2 is a different spike of K such that $\widetilde{N}_K(Q_1) \subseteq \widetilde{N}_K(Q_2)$, then Q_2 is a bound spike and, in particular, there exits $K' \in \mathcal{K}(G)$ distinct from K such that $N[Q_2] = K \cup K'$ and $\widetilde{N}_K(Q_1) \subseteq K'$.

(jj) If Q_1 and Q_2 are different non-trivial spikes of K, then there exist $v \in \widetilde{N}_K(Q_1) \setminus N(Q_2)$ and $w \in \widetilde{N}_K(Q_2) \setminus N(Q_1)$ that are non-adjacent.

PROOF. (j) Q_2 is not simplicial, since $\widetilde{N}_K(Q_2)$ is non-empty. Suppose now Q_2 is non-trivial, and pick $q_1 \in Q_1$, $q_2 \in Q_2$. Then $\widetilde{N}_K(q_2)$ and $\widetilde{N}_K(q_1)$ are complete to each other, since $\widetilde{N}_K(Q_2)$ is a clique, $\widetilde{N}_K(q_2) \subseteq \widetilde{N}_K(Q_2)$ and, by hypothesis, $\widetilde{N}_K(q_1) \subseteq \widetilde{N}_K(Q_1) \subseteq \widetilde{N}_K(Q_2)$. This implies that q_1, q_2 belong to the same spike of K, a contradiction. Thus, Q_2 is a non-trivial spike, and consequently $N[Q_2] = K \cup K'$ for some $K' \in \mathcal{K}(G)$. Moreover, since $\widetilde{N}_K(Q_1) \cap K = \emptyset$, $\widetilde{N}_K(Q_1) \subseteq K'$.
(jj) $\widetilde{N}_K(q_1) \cup \widetilde{N}_K(q_2)$ is not a clique, since q_1 and q_2 belong to different spikes. As $\widetilde{N}_K(q_1)$ and $\widetilde{N}_K(q_2)$ are cliques, there exist $v \in \widetilde{N}_K(q_1)$ and $w \in \widetilde{N}_K(q_2)$ that are non-adjacent. Moreover, since $\widetilde{N}(Q_1)$ is a clique, $w \notin \widetilde{N}(Q_1)$, that is, $w \notin N(Q_1)$. Similarly $v \notin N(Q_2)$. ■ □

CLAIM 5. *Let $Q \in \mathcal{Q}(K)$ be a non-trivial spike for some articulation clique $K \in \mathcal{K}(G)$, and let $K' \in \mathcal{K}(G), K' \neq K$, another articulation clique of G. If $\widetilde{N}_K(Q) \cap K' \neq \emptyset$, then $\widetilde{N}_K(Q) \cap K' \subseteq Q'$, for some spike $Q' \in \mathcal{Q}(K')$.*

PROOF. It follows from Lemma 4.4 that $Q \cap K' = \emptyset$, as Q is non-trivial spike and therefore does not intersect any other spike of $Q \in \mathcal{Q}(K)$, and so, in particular, does not intersect K'. Now suppose, by contradiction, that $\widetilde{N}_K(Q) \cap Q', \widetilde{N}_K(Q) \cap Q'' \neq \emptyset$, for some distinct $Q', Q'' \in \mathcal{Q}(K')$. We first argue that Q', Q'' are non-trivial spikes of K'. They are not simplicial spikes, since they are adjacent to Q, which we argued lie outside K'. Suppose now that Q'' is a bound spike; therefore there exists an articulation clique $K'' \in \mathcal{K}(G) \setminus K'$ such that $Q'' = K' \cap K''$ and $N[Q''] = K' \cup K''$. Note that $K'' \neq K$, as otherwise $Q'' \subseteq K$, while we are assuming that $\widetilde{N}_K(Q) \cap Q'' \neq \emptyset$. Also, since $Q \cap K' = \emptyset$, it follows that $N(Q'') \cap Q \subseteq K''$, and thus there is some vertex $v \in Q$ that belongs to K''. Then v is bound, as it belongs to K and K'', and therefore Q is a bound spike, a contradiction. Thus, both Q' and Q''

are non-trivial spikes. By Claim 4, there exists $u \in \widetilde{N}_{K'}(Q') \setminus N(Q'')$ and $v \in \widetilde{N}_{K'}(Q'') \setminus N(Q')$ such that $uv \notin E$. At least one between u and v does not belong to K, say without loss of generality u. As $Q \cap \widetilde{N}_{K'}(Q') \neq \emptyset$ and $\widetilde{N}_{K'}(Q')$ is a clique, then u is a vertex outside K that is adjacent to some vertex of Q, i.e., $u \in \widetilde{N}_K(Q)$. This leads to contradiction, since $\widetilde{N}_K(Q)$ is a clique and u is anti-complete to Q''. ■ □

CLAIM 6. *For each $K \in \mathcal{K}(G)$ and each non-trivial spike $Q \in \mathcal{Q}(K)$, we have that $N_{G|_{\mathcal{K}(G)}}(Q) = \widetilde{N}_K(Q)$ is a non-empty clique in $G|_{\mathcal{K}(G)}$.*

PROOF. Since Q is a non-trivial spike, by construction, for each $q \in Q$, $N_{G|_{\mathcal{K}(G)}}[q] = Q \cup \widetilde{N}_K(q)$. Therefore, $N_{G|_{\mathcal{K}(G)}}(Q) = \widetilde{N}_K(Q)$, and it is non-empty, as vertices in Q are non-simplicial. We also claim that $\widetilde{N}_K(Q)$ is a clique in G. Suppose to the contrary it is not: then there exists an articulation clique $K' \in \mathcal{K}(G)$, $K' \neq K$, and two distinct spikes $Q', Q'' \in \mathcal{Q}(K')$ such that $Q' \cap \widetilde{N}_K(Q)$, $Q'' \cap \widetilde{N}_K(Q) \neq \emptyset$. But this contradicts Claim 5. ■ □

CLAIM 7. *Let $K \in \mathcal{K}(G)$, Q_1, Q_2 different spikes of $\mathcal{Q}(K)$, and $v \in \widetilde{N}(Q_1) \cap \widetilde{N}(Q_2)$. Then for each covering of $N[v]$ with two maximal cliques H_1, H_2, we can assume without loss of generality that $Q_1 \cap N(v) \subseteq H_1$ and $Q_2 \cap N(v) \subseteq H_2$. Thus, in particular, v is adjacent to at most two spikes from $\mathcal{Q}(K)$. Moreover, if both Q_1 and Q_2 are non-trivial spikes, then $Q_1 \cap N(v) \subseteq H_1 \setminus H_2$ and $Q_2 \cap N(v) \subseteq H_2 \setminus H_1$.*

PROOF. First suppose that both spikes Q_1 and Q_2 are non-trivial: by Claim 4, there exist $z_1 \in \widetilde{N}(Q_1) \setminus N(Q_2)$ and $z_2 \in N(Q_2) \setminus \widetilde{N}(Q_1)$ such that $z_1 z_2 \notin E$. Note that $z_1, z_2 \in N(v)$. It follows that any covering of $N[v]$ with two maximal cliques H_1, H_2 is such that without loss of generality $(Q_1 \cap N(v)) \cup \{z_1\} \subseteq H_1 \setminus H_2$ and $(Q_2 \cap N(v)) \cup \{z_2\} \subseteq H_2 \setminus H_1$.

Now suppose that at least one of Q_1, Q_2 is a bound spike, say without loss of generality Q_1. Therefore, $Q_1 = K \cap K'$, where both K and K' are articulation cliques. As Q_1 is made of true twins, we have that v is complete to Q_1 and since $v \notin K$, then $v \in K'$. As K' is an articulation clique, there exists a maximal clique H_2 such that $\{K', H_2\}$ is the unique pair of maximal cliques covering $N[v]$. Note that each $w \in N(v) \cap Q_2$ does not belong to K' (else $w \in Q_1$). Thus $Q_1 \subseteq K'$, $Q_2 \cap N(v) \subseteq H_2$, and the result follows. ■ □

CLAIM 8. *Let v be a vertex of G that does not belong to any articulation clique. Then any pair of maximal cliques of G covering $N_G[v]$ is also a pair of maximal cliques of $G|_{\mathcal{K}(G)}$ covering $N_{G|_{\mathcal{K}(G)}}[v]$ and vice versa.*

PROOF. Let v be a vertex of $V(G)$ that does not belong to any articulation clique of G. Note that $N_{G|_{\mathcal{K}(G)}}[v] = N_G[v]$. Let $\{H_1, H_2\}$ be a pair of maximal cliques of G covering $N_G[v]$ (note that such a pair exists, since G is quasi-line). As $N_{G|_{\mathcal{K}(G)}}[v] = N_G[v]$, it follows that $H_1 \cup H_2$ covers $N_{G|_{\mathcal{K}(G)}}[v]$. We also claim that $H_1 \cup H_2$ are both maximal cliques of $G|_{\mathcal{K}(G)}$. First observe that H_1 is a clique: if the contrary there would be two non-trivial spikes Q_1, Q_2 of a same articulation clique such that $Q_1 \cap H_1 \neq \emptyset$ and $Q_2 \cap H_1 \neq \emptyset$. Note that Q_1, Q_2 would be both non-trivial spikes, as $v \in N(Q_1)$, $v \in N(Q_2)$ and v does not belong to any articulation clique of G. But then we would contradict Claim 7. So both H_1 and H_2 are cliques. Then they are maximal, since $E(G|_{\mathcal{K}(G)}) \subseteq E(G)$.

Now suppose vice versa that $\{H_1, H_2\}$ is a pair of maximal cliques of $G|_{\mathcal{K}(G)}$ covering $N_{G|_{\mathcal{K}(G)}}[v]$. Again, $H_1 \cup H_2$ covers $N_G[v]$. Also it is trivial that H_1 and H_2 are cliques of G, as $E(G|_{\mathcal{K}(G)}) \subseteq E(G)$. Suppose now that H_1 is not a maximal clique of G; as $N_{G|_{\mathcal{K}(G)}}[v] = N_G[v]$, it follows that there exists $z \in H_2 \setminus H_1$ such that $H_1 \subseteq N_G(z)$, while $H_1 \not\subseteq N_{\mathcal{K}(G)}(z)$. Therefore, there must exist some articulation clique $K \in \mathcal{K}(G)$ with two spikes $Q_1, Q_2 \in \mathcal{Q}(K)$ such that $z \in Q_2$ and $Q_1 \cap (H_1 \setminus H_2) \neq \emptyset$. Note that Q_1, Q_2 are both non-trivial spikes, as $z \in N(Q_1)$, $z \in N(Q_2)$ and z does not belong to any articulation clique of G. Note also that there must exist a pair $\{H_1', H_2'\}$ of maximal cliques of G covering $N_G[v]$ with $z \in H_1' \cap H_2'$. But then we are in contradiction with Claim 7. ■ □

CLAIM 9. *$G|_{\mathcal{K}(G)}$ is quasi-line.*

PROOF. We first show that each vertex is regular in $G|_{\mathcal{K}(G)}$. This is trivial for vertices of G that are either simplicial or bound. Now let $v \in V(G)$ be a vertex that is neither simplicial nor bound. First suppose that v belongs to some articulation clique $K \in \mathcal{K}(G)$, and let $Q \in \mathcal{Q}(K)$ be such that $v \in Q$; then $N_{G|_{\mathcal{K}(G)}}(v) \subseteq Q \cup \widetilde{N}_K(Q)$ and so v is regular, from Claim 6. Now suppose that v does not belong to any articulation clique of G; then v is regular, as G is quasi-line and Claim 8 holds. ■ □

CLAIM 10. *Let K be an articulation clique of G and $Q \in \mathcal{Q}(K)$ a non-trivial spike of K. If $Q \cup \widetilde{N}_K(Q)$ is a clique in G, then there exists an articulation clique $K_1 \in \mathcal{K}(G) \setminus K$ and a non-trivial spike $Q_1 \in \mathcal{Q}(K_1)$ such that $\widetilde{N}_K(Q) = Q_1$ and $\widetilde{N}_{K_1}(Q_1) = Q$.*

PROOF. It follows from Claim 6 that $N_{G|_{\mathcal{K}(G)}}(Q) = \widetilde{N}_K(Q)$ is a non-empty clique in $G|_{\mathcal{K}(G)}$. Therefore also $Q \cup N_{G|_{\mathcal{K}(G)}}(Q)$ is a clique in $G|_{\mathcal{K}}(G)$: no edge between Q and $N_{G|_{\mathcal{K}(G)}}(Q)$ is removed in the ungluing of the cliques in $\mathcal{K}(G)$, since Q is a non-trivial spike and so no vertex of Q belongs to an articulation clique $K' \neq K$. We now show that $Q \cup \widetilde{N}_K(Q)$ induces a component of $G|_{\mathcal{K}(G)}$.

Suppose first that there exists a vertex $w \in K \setminus Q$ that is complete to $\widetilde{N}_K(Q)$. By Claim 4, w is a bound vertex with $N[w] = K \cup K_1$, for some $K_1 \in \mathcal{K}(G) \setminus K$, and $\widetilde{N}_K(Q) \subseteq K_1$. Note that $\widetilde{N}_K(Q) \subseteq Q_1$, for some spike $Q_1 \in \mathcal{Q}(K_1)$, as we know that $\widetilde{N}_K(Q)$ is a clique in $G|_{\mathcal{K}(G)}$. We claim that Q_1 is a non-trivial spike. Indeed each vertex $v \in \widetilde{N}_K(Q) \subseteq Q_1$ is complete to Q and, in particular, $Q \subseteq \widetilde{N}_{K_1}(Q_1)$, as Q is non-trivial and so $Q \cap K_1 = \emptyset$. Therefore, no vertex $v \in \widetilde{N}_K(Q)$ is bound: if the contrary, $\{v\} \cup Q$ would belong to an articulation clique different from K, which again contradicts Q being non-trivial. Hence Q_1 is non-trivial, and, in particular, Q_1 does not intersect K.

We now argue that $\widetilde{N}_{K_1}(Q_1) = Q$. Since $\widetilde{N}_{K_1}(Q_1)$ is a clique of G, all vertices from $\widetilde{N}_{K_1}(Q_1) \setminus Q$ are complete to Q. But since $\widetilde{N}_K(Q) \subseteq Q_1$, it follows that $\widetilde{N}_{K_1}(Q_1) \setminus Q \subset K$. But then $\widetilde{N}_{K_1}(Q_1) \setminus Q = \emptyset$, i.e., $\widetilde{N}_{K_1}(Q_1) = Q$, else we would contradict Claim 5. Moreover, for each $t \in Q_1$, $\widetilde{N}_{K_1}(t)$ is non-empty and contained in Q, which implies $t \in \widetilde{N}_K(Q)$. Thus, $Q_1 = \widetilde{N}_K(Q)$. Summing up, we have that $N_{G|_{\mathcal{K}(G)}}(Q) = \widetilde{N}_K(Q)$, and $N_{G|_{\mathcal{K}(G)}}(\widetilde{N}_K(Q)) = N_{G|_{\mathcal{K}(G)}}(Q_1) = \widetilde{N}_{K_1}(Q_1) = Q$.

Thus, we can assume there exists no $w \in K \setminus Q$ that is complete to $\widetilde{N}_K(Q)$. We now prove that this case leads to contradiction, by showing that $Q \cup \widetilde{N}_K(Q)$ is an articulation clique in G, and thus Q is not a non-trivial spike. Note first that, in this case, $Q \cup \widetilde{N}_K(Q)$ is a maximal clique in G, and it is crucial for all vertices in Q (note that the vertices in Q are true twins, then the statement follows from Lemma 4.3). We now show $Q \cup \widetilde{N}_K(Q)$ is also crucial for all vertices from $\widetilde{N}_K(Q)$. For each $v \in \widetilde{N}_K(Q)$, let $T_v := N(v) \setminus (Q \cup \widetilde{N}_K(Q))$. If $T_v = \emptyset$, then v is simplicial and $Q \cup \widetilde{N}_K(Q)$ is crucial for v. So suppose that $T_v \neq \emptyset$, but, first, assume that $T_v \cap K = \emptyset$. In this case, T_v is anti-complete to Q, since $N[Q] = K \cup \widetilde{N}_K(Q)$. Applying Lemma 3.4, it follows that $Q \cup \widetilde{N}_K(Q)$ is crucial for v. Last, suppose that $T_v \cap K \neq \emptyset$. Let $S := (N(v) \cap K) \setminus Q$ (note $S \neq \emptyset$ by hypothesis). By Claim 7, $S \subseteq Q'$ for some $Q' \in \mathcal{Q}(K)$ distinct from Q, and for each bipartition H_1, H_2 of $N[v]$ into two maximal cliques, without loss of generality $Q \subseteq H_1, S \subseteq H_2$. Now, since we are assuming that no $w \in K \setminus Q$ is complete to $\widetilde{N}_K(Q)$, it follows, in particular, that no vertex in S is complete to $\widetilde{N}_K(Q)$, thus $\widetilde{N}_K(Q) \subset H_1$. Since $Q \cup \widetilde{N}_K(Q)$ is a maximal clique, it follows that $H_1 = Q \cup \widetilde{N}_K(Q)$ and so $Q \cup \widetilde{N}_K(Q)$ is crucial for v. Since $Q \cup \widetilde{N}_K(Q)$ is a maximal clique in G and it is crucial for all its vertices, it is an articulation clique of G, and this concludes the proof. ■ □

CLAIM 11. *$G|_{\mathcal{K}(G)}$ is net-free.*

PROOF. Suppose to the contrary that $G|_{\mathcal{K}(G)}$ is no net-free, and let K be a net clique. Therefore, there exist vertices $a_1, a_2, a_3 \in K$, $v_1, v_2, v_3 \notin K$ such that $a_i v_j \in E_{\mathcal{K}(G)}$ if and only if $i = j$, and $v_i v_j \notin E_{\mathcal{K}(G)}$ for each $i \neq j$. Recall (cfr. the proof of Lemma 3.10) that (S_1, S_2, S_3, S_4) is a partition of the vertices of K, for $S_i = N_{G|_{\mathcal{K}(G)}}(v_i) \cap K$, and $S_4 = K \setminus (S_1 \cup S_2 \cup S_3)$.

Since $E(G|_{\mathcal{K}(G)}) \subseteq E$, K is also a clique in G. In order to obtain a contradiction with the assumption that K is a net clique in $G|_{\mathcal{K}(G)}$, it suffices to show that K is an articulation clique in G, i.e., that it is maximal and crucial for all its vertices. The first part of the proof is devoted to show that K is a maximal clique of G. Suppose to the contrary that there exists a vertex w that is complete to K in G, while it is not complete to K in $G|_{\mathcal{K}(G)}$. This implies that there exist spikes $Q'_1 \neq Q'_2$ of some articulation clique $K' \in \mathcal{K}(G)$ such that $w \in Q'_2$, $Q'_1 \cap K \neq \emptyset$. This also implies that w is anti-complete to Q'_1 in $G|_{\mathcal{K}(G)}$ and that $K' \cap K \subseteq Q'_1$, otherwise K would not be a clique in $G|_{\mathcal{K}(G)}$. Since Q'_1 is a clique in $G|_{\mathcal{K}(G)}$ and $N_{G|_{\mathcal{K}(G)}}(Q'_1)$ is a clique in $G|_{\mathcal{K}(G)}$ (this is immediate if Q'_1 is either simplicial or bound, while it follows from Claim 6 if it is non-trivial), Q'_1 intersects at most one set among S_1, S_2, S_3. Thus, we can suppose $Q'_1 \cap (S_1 \cup S_2) = \emptyset$, which implies that $Q'_1 \cap K$ is anti-complete to $\{v_1, v_2\}$ in $G|_{\mathcal{K}(G)}$.

We now argue that w is complete to S_1 and S_2 in $G|_{\mathcal{K}(G)}$. Suppose to the contrary that without loss of generality it is not complete to S_1. Then $S'_1 = S_1 \setminus N_{G|_{\mathcal{K}(G)}}(w) \neq \emptyset$ and there exist spikes $Q''_1 \neq Q''_2$ of some articulation clique $K'' \in \mathcal{K}(G)$ such that $w \in Q''_2$ and $S'_1 \subseteq Q''_1$. Also, $K'' \cap K \subseteq Q''_1$ (in fact,

K does not intersect other spikes of K'', otherwise K would not be a clique of $G|_{\mathcal{K}(G)}$). Note that $K'' \neq K'$, as K'' intersects K in $S_1' \subseteq S_1$, while $K' \cap K \subseteq Q_1' \subseteq S_3 \cup S_4$. It follows that $w \in K'' \cap K'$ is a bound vertex with $N_G[w] = K'' \cup K'$. But again since Q_1'' and $N_{G|_{\mathcal{K}(G)}}(Q_1'')$ are cliques in $G|_{\mathcal{K}(G)}$, Q_1'' intersects at most one set among S_1, S_2, S_3 and thus this set is S_1. Therefore, as $N_G[w] = K'' \cup K'$, $K' \cap K \subseteq Q_1' \subseteq S_3 \cup S_4$ and $K'' \cap K \subseteq Q_1'' \subseteq S_1$, it follows that w is anti-complete to S_2 in G. This is a contradiction to the fact that w is complete to K in G. Therefore, w is complete to S_1 and S_2 in $G|_{\mathcal{K}(G)}$ and, in particular $w \neq v_1, v_2$.

Note now that w is adjacent to v_1, v_2 in $G|_{\mathcal{K}(G)}$ otherwise, for instance, $(a_1; v_1, w, q)$ is a claw in $G|_{\mathcal{K}(G)}$ for any $q \in Q_1' \cap K$, while $G|_{\mathcal{K}(G)}$ is quasi-line from Claim 9. Thus $wv_1, wv_2 \in E(G|_{\mathcal{K}(G)})$, while $v_1v_2 \notin E(G|_{\mathcal{K}(G)})$ by definition. Recall that w belongs to some spike Q_2' from K', K' being an articulation clique of $\mathcal{K}(G)$. Again, both Q_2' and $N_{G|_{\mathcal{K}(G)}}(Q_2')$ are cliques in $G|_{\mathcal{K}(G)}$. So either v_1 or $v_2 \in Q_2'$, but not both: say without loss of generality $v_2 \in Q_2'$. On the other hand, as $K' \cap K \subseteq Q_1'$, it follows that $Q_2' \cap K = \emptyset$ and thus $a_2 \notin Q_2'$. Therefore $\{a_2, v_1\}$ is a stable set of size 2 in $N_{G|_{\mathcal{K}(G)}}(Q_2')$, contradicting the fact that $N_{G|_{\mathcal{K}(G)}}(Q_2')$ is a clique.

Thus, K is maximal in G. In the remaining of the proof we show that K is crucial for all its vertices in G. Suppose not: then there exists $v \in K$ such that K is not crucial for v in G. However, v belongs to some articulation clique of G: otherwise, since K is crucial for v in $G|_{\mathcal{K}(G)}$, it follows from Claim 8 that K is crucial for v also in G. So let K_1 be an articulation clique of G containing v, and let $Q_1 \in \mathcal{Q}(K_1)$ be such that $v \in Q_1$. Trivially, as we are assuming that K is not an articulation clique of G, we have that $K_1 \neq K$. Moreover, $K_1 \cap K \subseteq Q_1$, else K is not a clique in $G|_{\mathcal{K}(G)}$. Note that $K \setminus Q_1 \subseteq \widetilde{N}_{K_1}(v)$. Suppose first that $K \setminus Q_1 = \widetilde{N}_{K_1}(v)$. Then $U[v] = K \cap K_1$ and, therefore, $\{K, K_1\}$ is the unique pair of maximal cliques covering $N_G[v]$. So, K is crucial for v in G, contradicting the assumptions.

Thus $\widetilde{N}_{K_1}(v) \setminus K \neq \emptyset$. We now show that $Q_1 = K \cap K_1$. By Claim 6, $\widetilde{N}_{K_1}(v)$ is a clique in $G|_{\mathcal{K}(G)}$, therefore, $\{\widetilde{N}_{K_1}[v], Q_1\}$ is a pair of cliques of $G|_{\mathcal{K}(G)}$ that cover $N_{G|_{\mathcal{K}(G)}}[v]$. Let $\{K', K''\}$ be a pair of *maximal* cliques of $G|_{\mathcal{K}(G)}$ that cover $N_{G|_{\mathcal{K}(G)}}[v]$ and are such that $\widetilde{N}_{K_1}[v] \subseteq K'$ and $Q_1 \subseteq K''$. Observe that since K is crucial for v in $G|_{\mathcal{K}(G)}$, one of K' or K'' is equal to K. But since we assumed $\widetilde{N}_{K_1}(v) \setminus K \neq \emptyset$ and K is maximal in G, it follows that $K = K''$ and thus $Q_1 \subseteq K$. As $K_1 \cap K \subseteq Q_1$, it follows that $Q_1 = K_1 \cap K$.

Let now $w \in \widetilde{N}_{K_1}(v) \setminus K$ and suppose that w is non-adjacent in $G|_{\mathcal{K}(G)}$ to some vertex $z \in N|_{\mathcal{K}(G)}(K)$. Since $\widetilde{N}_{K_1}(Q_1)$ is a clique and $Q_1 \subseteq K$, it follows that z is adjacent to some vertex $u \in K \setminus Q_1$. Also, $w \in N_{G|_{\mathcal{K}(G)}}(u)$, since $w, u \subseteq \widetilde{N}_{K_1}(Q_1)$ and the latter is a clique. But then K cannot be crucial for u in $G|_{\mathcal{K}(G)}$, since $w, z \in N_{G|_{\mathcal{K}(G)}}(u) \setminus K$ and $wz \notin E_{\mathcal{K}(G)}$. Hence w is complete to $N|_{\mathcal{K}(G)}(K)$ in $G|_{\mathcal{K}(G)}$. This implies that $w \neq v_1, v_2, v_3$ and that $(w; v_1, v_2, v_3)$ is a claw in $G|_{\mathcal{K}(G)}$, which cannot happen by Claim 9. This gives the required contradiction and concludes the proof. ■ □

(**i**) It follows from Claim 9.

(**ii**). Let $Q \in \mathcal{Q}(K)$ for some $K \in \mathcal{K}(G)$. By definition, Q entirely belongs to for some component C of $G|_{\mathcal{K}(G)}$. We now show that C is distance simplicial with respect to Q. If the component coincides with Q, the statement is trivial, thus suppose that Q has non-empty neighborhood in $G|_{\mathcal{K}(G)}$. This implies that Q is non-trivial. If $Q \cup N_{G|_{\mathcal{K}(G)}}(Q)$ is a clique, Claim 10 and the definition of ungluing imply that it is a clique-component of $G|_{\mathcal{K}(G)}$, and again the statement holds. So suppose $Q \cup N_{G|_{\mathcal{K}(G)}}(Q)$ is not a clique; as C is quasi-line, it is net-free by Claim 11, and $N_{G|_{\mathcal{K}(G)}}(Q)$ is a clique, by Claim 6. Then we use Lemma A.1 and conclude that the statement holds true.

The last part of the statement follows trivially by construction and Lemma 4.3.

(**iii**). Let $C \in \mathcal{C}$ be a component of $G|_{\mathcal{K}(G)}$ and suppose that two non-trivial spikes $Q_1, Q_2 \in \mathcal{Q}(\mathcal{K}(G))$ belong to C. Note that $Q_1 \cap Q_2 = \emptyset$: this is trivial if Q_1 and Q_2 belong to the same articulation clique, otherwise it follows from Lemma 4.4. We know form part (ii) that C is distance simplicial with respect to $Q_1$and Q_2. Let j_2 be the maximum integer such that $N_{j_2}(Q_1) \cap Q_2 \neq \emptyset$, where $N_j(Q_1)$ is the $j-$th neighborhood of Q_1 in C. As Q_2 is a clique, it follows that $Q_2 \subseteq N_{j_2}(Q_1) \cup N_{j_2-1}(Q_1)$. The statement will therefore directly follow from the next claim.

CLAIM 12. $N_{j_2+1}(Q_1) = \emptyset$.

PROOF. Recall that $N_{G|_{\mathcal{K}(G)}}(Q_2)$ is a clique. Suppose first $j_2 = 1$. Note that in this case $Q_2 \subseteq N_1(Q_1)$, as $Q_1 \cap Q_2 = \emptyset$. Suppose, by contradiction, that $N_2(Q_1) \neq \emptyset$, and let $w \in N_2(Q_1)$. Then w is anti-complete to Q_2 in $G|_{\mathcal{K}(G)}$, otherwise Q_2 would have neighbors from both Q_1 and $N_2(Q_1)$

in $G|_{\mathcal{K}(G)}$, contradicting the fact that $N_{G|_{\mathcal{K}(G)}}(Q_2)$ is a clique. Thus w is adjacent to some vertex $v \in N_1(Q_1) \setminus Q_2$.

We now show that $Q_1 \cup N_1(Q_1)$ is a clique. Suppose to the contrary it is not, then there exists a pair of vertices $s, t \in Q_1 \cup N_1(Q_1)$ that are non-adjacent in $G|_{\mathcal{K}(G)}$. As Q_1, $N_1(Q_1)$ are cliques in $G|_{\mathcal{K}(G)}$, we can assume without loss of generality $s \in Q_1$, $t \in N_1(Q_1)$. As $Q_1, N_1(Q_1) \setminus Q_2 \subseteq N_{G|_{\mathcal{K}(G)}}(Q_2)$ and the latter is a clique, Q_1 is complete to $N_1(Q_1) \setminus Q_2$. Thus, $t \in Q_2$. But then $(v; s, t, w)$ is a claw in $G|_{\mathcal{K}(G)}$, a contradiction to Claim 9. Thus, $N_{G|_{\mathcal{K}(G)}}[Q_1] = Q_1 \cup N_1(Q_1)$ is a clique. It follows from Claim 10 that $V(C) = N_{G|_{\mathcal{K}(G)}}[Q_1]$. This contradicts the fact that $N_2(Q_1) \neq \emptyset$.

Hence assume that $j_2 > 1$ and suppose to the contrary that $N_{j_2+1}(Q_1) \neq \emptyset$. Q_2 is anti-complete to $N_{j_2+1}(Q_1)$: else there would be a vertex v in $N_{G|_{\mathcal{K}(G)}}(Q_2) \cap (N_{j_2-1}(Q_1) \cup N_{j_2-2}(Q_1))$ and a vertex z in $N_{G|_{\mathcal{K}(G)}}(Q_2) \cap N_{j_2+1}(Q_1)$ that are non-adjacent. As $N_{j_2+1}(Q_1)$ is non-empty and anti-complete to Q_2, also $N_{j_2}(Q_1) \setminus Q_2$ is non-empty. Then we claim that $Q_2 \cap N_{j_2-1}(Q_1) = \emptyset$: if the contrary, then there would two non-adjacent neighbors of Q_2, respectively in $N_{j_2-2}(Q_1)$ and in $N_{j_2}(Q_1) \setminus Q_2$. But then $N_2(Q_2)$ picks two non adjacent vertices, respectively in $N_{j_2-2}(Q_1)$ and $N_{j_2+1}(Q_1)$, contradicting the fact that Q_2 is distance simplicial. ■ □

(**iv**). Let $C \in \mathcal{C}$ be a component of $G|_{\mathcal{K}(G)}$ and suppose to the contrary that $Q_1, \dots Q_l \in \mathcal{Q}(\mathcal{K}(G))$ belong to C, for some $l \geq 3$. It follows from the last statement of (ii) that the spikes $Q_1, \dots Q_l$ are non-trivial (recall that no vertex belongs to more than two articulation clique, and therefore no vertex belongs to more than two spikes); therefore they are pairwise-disjoint: this is trivial for pairs of spikes from a same articulation clique, otherwise it follows from Lemma 4.4. We also know form part (ii) that C is distance simplicial with respect to $Q_1, \dots Q_l$. Finally, for $i = 2, 3$, let j_i be the maximum integer such that $N_{j_i}(A_1) \cap Q_i \neq \emptyset$, where $N_j(Q_1)$ is the $j-$th neighborhood of Q_1 in $G|_{\mathcal{K}(G)}$. It then follows from part (iii) that $j_2 = j_3$ and that $N_{j_2+1}(Q_1) = \emptyset$.

We claim that $Q_2 \subseteq N_{j_2}(Q_1)$. That is trivial if $j_2 = 1$, since $Q_1 \cap Q_2 = \emptyset$. Hence assume that $j_2 > 1$, and suppose to the contrary that $Q_2 \cap N_{j_2-1}(Q_1) \neq \emptyset$. Since $N_{j_2-2}(Q_1) \cap N_{G|_{\mathcal{K}(G)}}(Q_2) \neq \emptyset$, it follows that the neighborhood of Q_2 contains a vertex from $N_{j_2-2}(Q_1)$ and a vertex from $Q_3 \cap N_{j_2}(Q_1)$, contradicting the fact that $N_{G|_{\mathcal{K}(G)}}(Q_2)$ is a clique.

Similarly, $Q_3 \subseteq N_{j_2}(Q_1)$. Thus, $Q_2 \cup Q_3$ is a clique, since $N_{j_2}(Q_1)$ is a clique. As $N_{G|_{\mathcal{K}(G)}}(Q_2)$, $N_{G|_{\mathcal{K}(G)}}(Q_3)$ are cliques, and Q_2 is complete to Q_3, it follows that Q_2 is complete to $N_{G|_{\mathcal{K}(G)}}(Q_2)$, i.e., $N_{G|_{\mathcal{K}(G)}}[Q_2]$ is a clique. Therefore, it follows from Claim 10 that $V(C) = N_{G|_{\mathcal{K}(G)}}[Q_2]$, and therefore C is a clique. We now derive a contradiction to this statement.

First observe that $Q_1, Q_2, ..., Q_l$ are spikes from different articulation cliques $K_1, K_2, ..., K_l \in \mathcal{K}(G)$, else C would not be a clique. We now show that $Q_{l+1} := V(C) \setminus (\cup_{i=1,\dots,l} Q_i)$ is empty. Otherwise, pick $v \in Q_{l+1}$. By definition, v does not belong to any articulation clique of G: then $N_G[v] = N_{G|_{\mathcal{K}(G)}}[v] = \cup_{i=1,\dots,l+1} Q_i = V(C)$, thus v is simplicial in G (recall that $E(G|_{\mathcal{K}(G)}) \subseteq E(G)$), a contradiction to Lemma 3.9. Therefore, $Q_{l+1} = \emptyset$.

Let $v \in Q_i$, for $i \in \{1, ..., l\}$. Observe that $\widetilde{N}_{K_i}(v) = \cup_{j=1,\dots,l; j\neq i} Q_j$. Then, it follows from Lemma 4.3 that the unique covering of $N_G[v]$ into two maximal cliques is given by $\{K_i, \cup_{i=1,\dots,l} Q_i\}$. Thus, $\cup_{i=1,\dots,l} Q_i$ is crucial for every vertex in it, and therefore $\cup_{i=1,\dots,l} Q_i$ is an articulation clique of G. Then the vertices of $\cup_{i=1,\dots,l} Q_i$ are bound, a contradiction.

(**v**). First observe that the set of strips $\{(C, \mathcal{A}(C)), C \in \mathcal{C}\}$ is well-defined, since by part (iv) for each $C \in \mathcal{C}$, $\mathcal{A}(C)$ is a multi-set with one or two cliques. Let G' be the graph obtained by composing $\{(C, \mathcal{A}(C)), C \in \mathcal{C}(G|_{\mathcal{K}(G)})\}$ with respect to the partition $\mathcal{P}$ that puts two extremities in the same class if and only if they are spikes from a same articulation clique. By definition of ungluing, $\cup\{V(C) : C \in \mathcal{C}\}$ partitions V, thus $V(G) = V(G')$. By definition of composition, two vertices u, v of G' are adjacent if and only if $uv \in E(C)$ for some $C \in \mathcal{C}$, or $u \in A_1, v \in A_2$ and A_1, A_2 both belong to some class of the partition $\mathcal{P}$. By the definition of $\mathcal{P}$, this implies that $uv \in E(G')$ if and only if $uv \in E(G)$. Thus $G' = G$ and we conclude the proof.

B. THE PROOF OF LEMMA 5.4

This section is devoted to the proof of Lemma 5.4. We will often refer to the following:

LEMMA B.1. [Lovász and Plummer 1986] *Let $G(V, E)$ be a claw-free graph with an induced 5-wheel centered in $a \in V$. Then $\alpha(a \cup N(a) \cup N_2(a)) \leq 3$.*

We now move to the proof of Lemma 5.3.

PROOF. We postpone the complexity issues to the end of the proof, and start by showing that each graph G that fulfills the hypothesis, satisfies conditions (i) or (ii) of the statement. In order to do that, we have to gather some more information on the structure of G. In the following, we denote the 5-wheel centered in a by $W = (a; u_1, u_2, u_3, u_4, u_5)$. Also, for $i \in [5]$, we denote by S_i the set of vertices in $N_2(a)$ whose adjacent vertices in W are exactly u_i, u_{i+1} (where we identify u_6 with u_1) and such that they either have a neighbor in $N_3(a)$, or they are simplicial. We also let $\widetilde{N_2}(a)$ be the set of vertices in $N_2(a) \setminus \bigcup_{i=1..5} S_i$.

We now investigate some properties of the graph G in the first three neighborhoods of the irregular vertex a.

CLAIM 13. *Let v be a vertex of $N_2(a)$. The following statements hold:*

(i) *The vertices of $\{u_1, u_2, u_3, u_4, u_5\}$ that are adjacent to v are at least two and they have consecutive indices.*
(ii) *If v has a neighbor in $N_3(a)$ or is simplicial, then v has exactly two neighbors in $\{u_1, u_2, u_3, u_4, u_5\}$, and they have consecutive indices.*

PROOF. We first prove that v has at least one neighbor in $\{u_1, u_2, u_3, u_4, u_5\}$. By contradiction, suppose there exists $v \in N_2(a)$ that is anti-complete to $\{u_1, u_2, u_3, u_4, u_5\}$. Since $v \in N_2(a)$, there exists $u \notin W$ such that $au \in E$ and $uv \in E$. Such a u must be adjacent to at least three consecutive vertices in $\{u_1, u_2, u_3, u_4, u_5\}$, otherwise there would exist a claw centered in a and picking u and two non-adjacent vertices. Thus without loss of generality let $uu_1 \in E$, $uu_2 \in E$, $uu_3 \in E$. But then there is a claw: $(u; u_1, v, u_3)$.

Now observe that if v is adjacent to some vertex in $\{u_1, u_2, u_3, u_4, u_5\}$, say u_1, then it is adjacent to u_5 or u_2 too, otherwise there would exist a claw: $(u_1; v, u_2, u_5)$. Statement (i) easily follows.

Now suppose that v has a neighbor $x \in N_3(a)$. Observe that v cannot be adjacent to two non-adjacent vertices in $\{u_1, u_2, u_3, u_4, u_5\}$, say u_1 and u_3, otherwise there would exist a claw: $(v; x, u_1, u_3)$. It follows that v has exactly two neighbors in $\{u_1, u_2, u_3, u_4, u_5\}$, and they have consecutive indices. Similarly if v is simplicial it cannot be adjacent to two non-adjacent vertices in $\{u_1, u_2, u_3, u_4, u_5\}$ and thus it follows that v has exactly two neighbors in $\{u_1, u_2, u_3, u_4, u_5\}$, and they have consecutive indices □

From Claim 13, it follows that the only vertices from $N_2(a)$ with an adjacent in $N_3(a)$ are those from $\bigcup_{i=1..5} S_i$.

CLAIM 14. *If v is a simplicial vertex in a claw-free graph G, and G has an induced $5-$wheel W with center a, then $v \notin \{a\} \cup N(a) \cup \widetilde{N_2}(a)$.*

PROOF. First observe that all vertices of a $5-$wheel are non-simplicial. Now let $u \in N(a) \setminus W$. In order to prevent claws, u is adjacent to two non-consecutive vertices in the $5-$wheel, and thus it is not simplicial. Last, take a simplicial vertex $v \in N_2(a)$; since it has to be adjacent to at least two vertices from $u_1, \ldots, u_5$ by Claim 13, in order to be simplicial it must be adjacent to exactly two consecutive vertices from $u_1, \ldots, u_5$, say u_1, u_2. Then, by definition, $v \in S_1$, which implies $v \notin \widetilde{N_2}(a)$. □

CLAIM 15. *If $\bigcup_{i=1..5} S_i = \emptyset$, we are in case (i) of the statement.*

PROOF. In this case, $V = \{a\} \cup N(a) \cup \widetilde{N_2}(a)$. By Claim 14, G has no simplicial vertices; By Lemma B.1, G has stability number at most 3. □

Thus, in the following, we can suppose that $\bigcup_{i=1,\ldots,5} S_i \neq \emptyset$.

CLAIM 16. *For $i = 1, 2, \ldots, 5$, the set $S_i \cup S_{i+1}$ is a clique.*

PROOF. Suppose the contrary, that is, there exist $x, y \in S_i \cup S_{i+1}$ that are not adjacent. Then, there would be the claw: $(u_{i+1}; a, x, y)$. □

CLAIM 17. *For $i = 1, \ldots, 5$, the set $S_i \cup (N(S_i) \cap (N(a) \cup \widetilde{N_2}(a)))$ is a clique.*

PROOF. without loss of generality we prove this claim for S_1 (we can assume $S_1 \neq \emptyset$ otherwise it is trivial). For sake of shortness, let $Q = N(S_1) \cap (N(a) \cup \widetilde{N_2}(a))$. We know from the above claim that S_1 is a clique. We now show that every vertex in S_1 is complete to Q. Suppose the contrary: then there exist $x \in N(a) \cup \widetilde{N_2}(a)$, $x \neq u_1, u_2$, and $y, z \in S_1$ such that $xy \in E$ and $xz \notin E$. As y is non-simplicial ($z, x \in n(y)$ and $zx \notin E$), it has a neighbor in $N_3(a)$, say w. Observe that $wx, wu_1, wu_2 \notin E$, therefore x must be adjacent to u_1 and u_2: else, say $xu_1 \notin E$, there would be the claw $(y; x, u_1, w)$. Moreover, in order to avoid the claws $(u_1; u_5, x, z)$ and $(u_2; u_3, x, z)$, it follows that u_5x and $u_3x \in E$. But then $(x; u_3, u_5, y)$ is a claw.

Finally we show that Q is a clique. Suppose the contrary. There exists $v, x \in Q$ that are not adjacent. We have just shown that S_1 is complete to Q, thus let $y \in S_1$, and we have that xy and $vy \in E$. As y is non-simplicial, there exists a vertex w of $N_3(a)$ that is adjacent to y, then there is the claw $(y; x, v, w)$. □

CLAIM 18. *Let $s \in S_i$ for some $i \in \{1, .., 5\}$. $N_3(a) \cap N(s)$ is a clique.*

PROOF. Suppose there exists $x, y \in N_3(a) \cap N(s)$ with $xy \notin E$. Let $z \in N(s) \cap N(a)$, then $(s; x, y, z)$ is a claw. □

CLAIM 19. *Let $s \in S_i$ and $t \in S_j$ for some $i \neq j \in \{1, .., 5\}$. If $st \in E$, then $N_3(a) \cap N(s) = N_3(a) \cap N(t)$.*

PROOF. Suppose that there exists $x \in N_3(a) \cap N(s)$ and $xt \notin E$. Since $i \neq j$, there exists $y \in \{u_1, ..., u_5\}$ such that $y \in N(s) \setminus N(t)$. But then $(s; x, y, t)$ is a claw. □

CLAIM 20. *Let $\mathcal{S}$ be the union of at least two non-empty subsets S_i. If $\mathcal{S}$ is a clique, then $\mathcal{S} \cup (N_3(a) \cap N(\mathcal{S}))$ is a clique.*

PROOF. Suppose that $S_i \cup S_j \subseteq \mathcal{S}$, $i \neq j$. For all $s \in S_i$, $t \in S_j$, $i \neq j$, $N_3(a) \cap N(s) = N_3(a) \cap N(t)$ by Claim 19. If we iterate this argument, we can conclude that each vertex $s \in \mathcal{S}$ has the same neighbors in $N_3(a)$. Finally, by Claim 18, $N_3(a) \cap N(s)$ is a clique and therefore $\mathcal{S} \cup (N_3(a) \cap N(\mathcal{S}))$ is a clique. □

We are now ready to prove our statements. Note that, by hypothesis and because of the properties of sets S_i shown above, we are in exactly one of the following cases.

(1) There is a single set $S_1, \dots, S_5$ that is non-empty.
(2) The set $\bigcup_{i=1..5} S_i$ is not a clique and the sets $S_1, \dots, S_5$ that are non-empty are two and non-consecutive.
(3) The set $\bigcup_{i=1..5} S_i$ is not a clique and the sets $S_1, \dots, S_5$ that are non-empty are three and non-consecutive.
(4) The sets $\bigcup_{i=1..5} S_i$ is a clique and the sets $S_1, \dots, S_5$ that are non-empty are at least two.
(5) The sets $\bigcup_{i=1..5} S_i$ is not a clique, and the sets S_i that are non-empty are consecutive, at least three.

We are now going to show that in cases $1-4$, we satisfy condition (ii) of the statement, while in case 5 we satisfy condition (i) of the statement. More precisely, we show that if case 5 holds, then $\alpha(G) \leq 3$ and $Simp(G) = \emptyset$, and that if either of cases $1-4$ hold, then there exists a strip $(H, \mathcal{A})$, that is either a 1-strip or a 2-strip with vertex disjoint extremities, such that H is an induced subgraph of G and the following properties hold (as usual, for $A \in \mathcal{A}$, we let $K(A) = N(A) \setminus V(H)$):

(j) $C(H, \mathcal{A})$ is anti-complete to $V \setminus V(H)$;
(jj) for $A \in \mathcal{A}$, $A \cup K(A)$ is an articulation clique of G;
(jjj) $a \in V(H)$ and $\alpha(H) \leq 3$;
(jv) $Simp(G) \cap V(H) = \emptyset$.

Let us consider case 1. Assume without loss of generality that $S_1 \neq \emptyset$. In this case, we set $H = G[\{a\} \cup N(a) \cup \widetilde{N_2}(a)]$, $A_1 = N(S_1) \cap (N(a) \cup \widetilde{N_2}(a))$ and $\mathcal{A} = \{A_1\}$; note that A_1 is non-empty and is a clique, following Claim 17. By hypothesis, $N_2(a) = \widetilde{N_2}(a) \cup S_1$ and, following Claim 13, $\widetilde{N_2}(a)$ is anti-complete to $N_3(a)$. Then, (j) holds by construction, (jjj) holds by construction and Lemma B.1, (jv) holds by construction and Claim 14. We are left with showing (jj); note that $K(A_1) = S_1$. Now observe that $A_1 \cup S_1$ is a clique, because of Claim 17, and it is maximal by construction. If $A_1 \cup S_1$ contains a simplicial vertex, then it is an articulation clique by Lemma 14, thus suppose it has none. Then each vertex of S_1 has an adjacent in $N_3(a)$ and $N_3(a)$ is anti-complete to A_1, thus it follows from Lemma 3.4 that every vertex in S_1 is strongly regular and $A_1 \cup S_1$ is crucial for these vertices. Now fix $v \in A_1$; as v is not simplicial, it has a neighbor w not in A_1, which is by construction anti-complete to S_1; it follows again from Lemma 3.4 that $A_1 \cup S_1$ is crucial for v (that is strongly regular). It follows that $A_1 \cup S_1$ is an articulation clique, as it is crucial for all its vertices.

Let us consider case 2. Assume without loss of generality that $S_1, S_3 \neq \emptyset$, and let $Q = N(S_1) \cap N(S_3) \cap \widetilde{N_2}(a)$. We set $H = G[\{a\} \cup N(a) \cup \widetilde{N_2}(a) \setminus Q]$, $A_1 = (N(S_1) \cap (N(a) \cup \widetilde{N_2}(a))) \setminus Q$, $A_2 = (N(S_3) \cap (N(a) \cup \widetilde{N_2}(a))) \setminus Q$ and $\mathcal{A} = \{A_1, A_2\}$; note that A_1 and A_2 are non-empty and cliques, following Claim 17: we know show that they are vertex disjoint. Let $s_1 \in S_1$, $s_3 \in S_3$ be a pair of non-adjacent vertices. Observe that $N(a) \cap N(S_1) \cap N(S_3) = \emptyset$, since any vertex from this set, say v, would be the center of the claw $(v; s_1, s_3, a)$. It follows that A_1 and A_2 are disjoint.

We now prove that statements $(j) - (jv)$ hold. (jjj) holds by Lemma B.1, (jv) holds by Claim 14. As for (j), note that, by hypothesis, $N_2(a) = \widetilde{N_2}(a) \cup S_1 \cup S_3$ and, following Claim 13, $\widetilde{N_2}(a)$ is anti-complete to $N_3(a)$. Therefore, by construction, if (j) does not hold, then there must be vertex v in Q that is adjacent to some vertex $w \in C(H, \mathcal{A})$. Following Claim 17, w is adjacent to both s_1 and s_3, and, by construction, s_1 and s_3 are anti-complete to $C(H, \mathcal{A})$: thus, $(v; w, s_1, s_3)$ is a claw, a contradiction. We are left to show (jj). Note that $K(A_1) = S_1 \cup Q$ and $K(A_2) = S_3 \cup Q$. First observe that $A_1 \cup K(A_1)$ and $A_2 \cup K(A_2)$ are cliques, because of Claim 17, and they are maximal by construction. We are left to show that they are articulation cliques. We show the statement for $A_1 \cup K(A_1)$, since the same argument holds for $A_2 \cup K(A_2)$. Note that, from Lemma 14, we may assume that no vertex of $A_1 \cup K(A_1)$ is simplicial.

The proof builds on Lemma 3.4 and Lemma 3.5, as we show that each vertex in $A_1 \cup K(A_1)$ is strongly regular and, in fact, $A_1 \cup K(A_1)$ is crucial for it. First we deal with a vertex $q \in Q$. Note that $N[q] = A_1 \cup S_1 \cup Q \cup A_2 \cup S_3$: in fact, $N(Q) \cap N_3(a) = \emptyset$, as $Q \subseteq \widetilde{N_2}(a)$, moreover, Q is anti-complete to $C(H, \mathcal{A})$, as we just observed. In particular, all vertices in Q are true twins. Now observe that S_1 is non-complete to S_3 by hypothesis; moreover, we already observed that S_1 is anti-complete to A_2 and S_3 is anti-complete to A_1. Therefore, following Lemma 3.5, with $X_1 = S_1$, $X_2 = A_1$, $Y_1 = S_3$, $Y_2 = A_2$, v is strongly regular and $A_1 \cup K(A_1) = A_1 \cup S_1 \cup Q$ is crucial for v. Next, pick $v \in S_1$; recall that we are assuming that v is not simplicial. Then note that $N(v) \setminus (A_1 \cup K(A_1)) \subseteq (S_3 \cup N_3(a))$, which implies that $N(v) \setminus (A_1 \cup K(A_1))$ is anti-complete to u_1; thus $A_1 \cup K(A_1)$ is crucial for v by Lemma 3.4. Now pick $v \in (A_1 \cup K(A_1)) \setminus (Q \cup S_1)$; it follows that $v \in N(a) \cup \widetilde{N_2}(a)$. As v is not simplicial, it follows that $N(v) \setminus (A_1 \cup K(A_1))$ is non-empty, and it is anti-complete to S_1 by construction. Then again, $A_1 \cup K(A_1)$ is crucial for v by Lemma 3.4.

Let us consider case 3. Assume without loss of generality that $S_1, S_2, S_4 \neq \emptyset$. We set $H = G[\{a\} \cup N(a) \cup \widetilde{N_2}(a) \cup S_1 \cup S_2]$, $A_1 = S_1 \cup S_2$, $A_2 = N(S_4) \cap (N(a) \cup \widetilde{N_2}(a)))$ and $\mathcal{A} = \{A_1, A_2\}$; note that A_1 and A_2 are cliques, following Claim 16 and Claim 17, and they are vertex disjoint. Before showing that statements $(j) - (jv)$ hold, we observe a few facts. First, each vertex in S_4 is either complete or anti-complete to $S_1 \cup S_2$. Indeed, suppose that $s_4 \in S_4$ has a non-adjacent in $S_1 \cup S_2$, say without loss of generality $s_1 \in S_1$. It follows that s_4 is anti-complete to S_2, otherwise there exists $s_2 \in S_2 \cap N(s_4)$ and $(s_2; s_4, s_1, u_3)$ is a claw. Applying a similar reasoning, s_4 is anti-complete to S_1. Thus we can partition S_4 in the classes $\{\bar{S}_4, \bar{\bar{S}}_4\}$, where the vertices in $\bar{S}_4$ (resp., in $\bar{\bar{S}}_4$) are those complete (resp., anti-complete) to $S_1 \cup S_2$. Note that $\bar{S}_4$ may be empty, while $\bar{\bar{S}}_4$ is not by hypothesis; moreover, they are both cliques by Claim 16. Note that the vertices in $S_1 \cup S_2$ are not simplicial, and therefore, by definition, have neighbors in $N_3(a)$. Let $T = S_1 \cup S_2 \cup \bar{S}_4$ and $Q = N(T) \cap N_3(a)$. Therefore, Q is a non-empty clique, moreover Claims 18 and 19 imply that $T \cup Q$ is also a clique.

Statement (j) holds by construction, statement (jjj) holds by Lemma B.1, (jv) holds by Claim 14 and because no vertex in $S_1 \cup S_2$ is simplicial. We are left to show (jj). Note that $K(A_1) = \bar{S}_4 \cup Q$ and $K(A_2) = S_4$. We already observed that $A_1 \cup K(A_1) = S_1 \cup S_2 \cup \bar{S}_4 \cup Q = T \cup Q$ is a clique. Also $A_2 \cup K(A_2)$ is a clique, following Claim 17. Moreover, by construction, they are both maximal.

We first show that $A_1 \cup K(A_1)$ is an articulation clique. As usual, we assume that no vertex in $A_1 \cup K(A_1)$ is simplicial (else the statement follows from Lemma 3.9) and show that each vertex in $A_1 \cup K(A_1)$ is strongly regular and, in fact, $A_1 \cup K(A_1)$ is crucial for it. We start with a vertex $v \in S_1$, and note that, in this case, $N(v) \setminus (A_1 \cup K(A_1))$ is contained in $N(a) \cup \widetilde{N_2}(a)$, and thus it is anti-complete to Q, that is non-empty. Thus, following Lemma 3.4, $A_1 \cup K(A_1)$ is crucial for v. Similarly for $v \in S_2$. Now we take a vertex v in Q; observe that $N(v) \setminus (A_1 \cup K(A_1)) \subseteq N_4(a) \cup (N_3(a) \setminus Q) \cup \bar{\bar{S}}_4$, which implies that $N(v) \setminus (A_1 \cup K(A_1))$, that is non-empty, as v is not simplicial, is anti-complete to $S_1 \cup S_2$: again, $A_1 \cup K(A_1)$ is crucial for v, following Lemma 3.4. Finally, we take a vertex v in $\bar{S}_4$. Note that $N[v] = Q \cup S_1 \cup S_2 \cup \bar{\bar{S}}_4 \cup A_2$, and that all vertices in $\bar{S}_4$ are true twins. Therefore, following Lemma 3.5, with $X_1 = A_1$, $X_2 = Q$, $Y_1 = N(S_4) \cap (N(a) \cup \widetilde{N_2}(a))$, $Y_2 = \bar{\bar{S}}_4$, v is strongly regular and $A_1 \cup K(A_1) = A_1 \cup Q \cup \bar{S}_4$ is crucial for v.

We now show that $A_2 \cup K(A_2)$ is an articulation clique: again, we use Lemma 3.4 and Lemma 3.5 for the proof. As usual, we assume that each vertex of $A_2 \cup K(A_2)$ is not simplicial, and therefore has a neighbor outside $A_2 \cup K(A_2)$. For a vertex in $\bar{S}_4$, define X_1, X_2, Y_1 and Y_2 as above: it follows from Lemma 3.5 that $Y_1 \cup Y_2 \cup U[v] = S_4 \cup (N(S_4) \cap (N(a) \cup \widetilde{N_2}(a))) = A_2 \cup K(A_2)$ is crucial for v. Now take a vertex v in A_2: each neighbor of v that is not in $A_2 \cup K(A_2)$ is anti-complete to $\bar{\bar{S}}_4$, so $A_2 \cup K(A_2)$ is crucial for v by Lemma 3.4. Take a vertex v in $\bar{\bar{S}}_4$: each neighbor of v that is not in $A_2 \cup K(A_2)$ belongs to $N_3(a)$, and therefore it is anti-complete to $u_4 \in A_2$: again, $A_2 \cup K(A_2)$ is crucial for v by Lemma 3.4.

Let us consider case 3. Assume without loss of generality that $S_1, S_2, S_4 \neq \emptyset$. We set $H = G[\{a\} \cup N(a) \cup \widetilde{N_2}(a) \cup S_1 \cup S_2]$, $A_1 = S_1 \cup S_2$, $A_2 = N(S_4) \cap (N(a) \cup \widetilde{N_2}(a)))$ and $\mathcal{A} = \{A_1, A_2\}$; note that A_1 and A_2 are cliques, following Claim 16 and Claim 17, and they are vertex disjoint. Before showing that statements $(j) - (jv)$ hold, we observe a few facts.

Let us consider case 4. In this case, we set $H = G[\{a\} \cup N(a) \cup N_2(a)]$, $A_1 = \bigcup_{i=1..5} S_i$ and $\mathcal{A} = \{A_1\}$; note that A_1 is a clique, by hypothesis. Note also that each non-empty S_i is made of non-simplicial vertices, and therefore, by definition, each vertex in some non-empty S_i has a neighbor in $N_3(a)$. By Claim 20, it also follows that $N_3(a)$ is a clique and it is complete to $\bigcup_{i=1..5} S_i$. (j) holds by construction, (jjj) by Lemma B.1, (jv) by Claim 14 and because each S_i is made of non-simplicial vertices. We are left with statement (jj). Observe that $K(A_1) = N_3(a)$. We already argued that $A_1 \cup K(A_1)$ is a clique, also it is maximal by construction. As usual, assume that $A_1 \cup K(A_1)$ has no simplicial vertex. Each vertex in $K(A_1)$ has a neighbor in $N_4(a)$, which is by definition anti-complete to A_1; each vertex in A_1 has a neighbor in $N(a) \cup \widetilde{N_2}(a)$ that is anti-complete to $K(A_1)$. Therefore, it follows from Lemma 3.4 that $A_1 \cup K(A_1)$ is an articulation clique.

Let us now consider case 5. Assume without loss of generality that $S_1, S_2, \ldots, S_k$, $k \geq 3$, are non-empty, with either $k = 5$ or $S_{k+1} = \emptyset$. As we already mentioned, we are going to show that $\alpha(G) \leq 3$ and $Simp(G) = \emptyset$. We first show that $\alpha(G) \leq 3$. By iteratively applying Claims 16 and 20, it follows that $N_3(a)$ is complete to $S_1 \cup S_2 \cup \ldots \cup S_k$, and that $N_3(a)$ is a clique. Now observe that $N_4(a) = \emptyset$. In fact, otherwise let z be a vertex of $N_3(a)$ that has some adjacent $w \in N_4(a)$. By hypothesis, there exist $x, y \in S_1 \cup S_2 \cup \ldots \cup S_k$ that are not adjacent, then there would be the claw $(z; w, x, y)$. Let s_1, s_2, s_3 be vertices in respectively S_1, S_2, S_3. We now show that $\widetilde{N_2}(a) = \emptyset$. Suppose the contrary and let $z \in \widetilde{N_2}(a)$. Observe that $\{u_1, u_2, u_3, u_4\} \subseteq N(S_1) \cup N(S_2) \cup N(S_3)$. On the other hand, $\{u_1, u_2, u_3, u_4\} \cap N(z)$ is non-empty from part (*i*) of Claim 13. It is a routine to check that then $\{u_1, u_2, u_3, u_4, s_1, s_2, s_3\} \subseteq N(z)$. In fact, suppose e.g. that $u_1 z \in E$: then $s_1 z \in E$ in order to avoid the claw $(u_1; a, s_1, z)$ and $u_2 z \in E$ in order to avoid the claw $(s_1; u_2, z, w)$, where $w \in N_3(a)$ is adjacent to s_1. If we iterate this argument, we can show that indeed $\{u_1, u_2, u_3, u_4, s_1, s_2, s_3\} \subseteq N(z)$. But this leads to a contradiction, since $(z; u_1, u_4, s_2)$ is a claw. Hence, $\widetilde{N_2}(a) = \emptyset$ and $N_2(a) = S_1 \cup S_2 \cup \ldots \cup S_k$, and therefore $N_2(a)$ is complete to $N_3(a)$. We know from Lemma B.1 that $\alpha(G[\{a\} \cup N(a) \cup N_2(a)]) \leq 3$. If $\alpha(G) \geq 4$, then there must exist a stable set S of size 4 picking exactly one vertex in $N_3(a)$ (since $N_3(a)$ is a clique and we showed $N_4(a) = \emptyset$). It follows that $|S \cap (\{a\} \cup N(a))| = 3$, which is a contradiction, as G is claw-free.

We are left to show that no vertex in G is simplicial. Recall that in this case $V = \{a\} \cup N(a) \cup (\bigcup_{i=1..5} S_i) \cup N_3(a)$. No vertex of $\{a\} \cup N(a)$ is simplicial by Claim 14. No vertex of $\bigcup_{i=1..5} S_i$ is simplicial, since we already argued that each vertex of $\bigcup_{i=1..5} S_i$ has a neighbor in $N_3(a)$. Last, observe that no vertex in $N_3(a)$ is simplicial, since $N_3(a)$ is complete to $\bigcup_{i=1..5} S_i$, that is not a clique by hypothesis.

We now move to complexity issues; let $n = |V|$ and $m = |E|$. We can compute the sets $N_j(a)$ for $j = 1, 2, 3, 4$ in time $O(m)$. While computing those sets, without extra calculation time we can record for each vertex $v \in N_2(a)$: which vertex from W it is adjacent to; if it has a neighbor in $N_3(a)$; if it is simplicial (recall that we are given the set $Simp(G)$). By definition, for $i = 1..5$, S_i is the subset of $N_2(a)$ formed by those vertices $a)$ whose neighbors in W are exactly u_i and u_{i+1}, and such that $b1)$ they have a neighbor in $N_3(a)$, or $b2)$ they are simplicial. Given a vertex of $N_2(a)$, we can check conditions $a)$, $b1)$, and $b2)$ in constant time from what argued above. Thus in $O(m)$-time we can build sets $S_1, \ldots, S_5$ and $\widetilde{N_2}(a)$. If $\bigcup_{i=1,\ldots,5} S_i = \emptyset$, from Claim 15 we are in case (i), thus we can suppose $\bigcup_{i=1..5} S_i \neq \emptyset$. Now observe that each clique of G has $O(\sqrt{m})$ vertices, since every vertex of a claw-free graph has at most $2\sqrt{m}$ neighbors (see Fact 3.2). Thus we can check in $O(m)-$time for each pair $\{i, j\}$ if $S_i \cup S_j$ is a clique. Then we can distinguish between cases $1 - 5$ in time $O(m)$. It is easy to check that, in each of the cases $1 - 5$, the strip $(H, \mathcal{A})$ and the sets $K(A)$ for each $A \in \mathcal{A}$ can be constructed in time $O(m)$.

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