Maximum weight stable set in (P_6, bull)-free graphs

Lucas Pastor

Mercredi 16 novembre 2016

Joint-work with **Frédéric Maffray**

Maximum weight stable set

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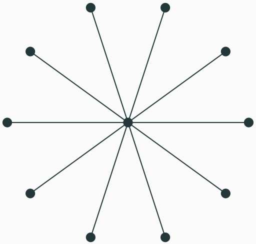

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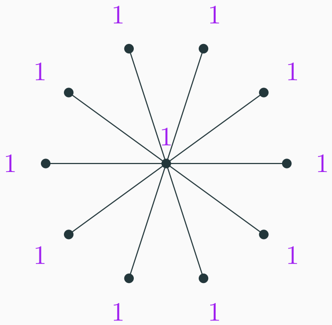

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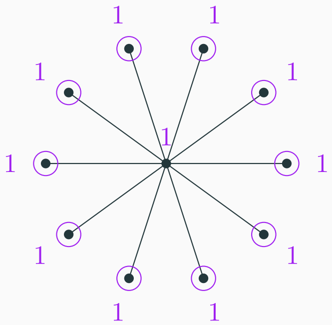

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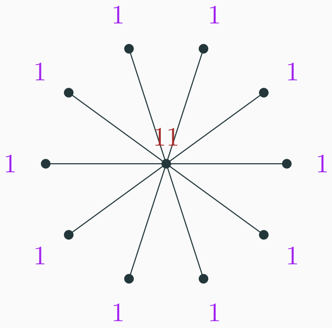

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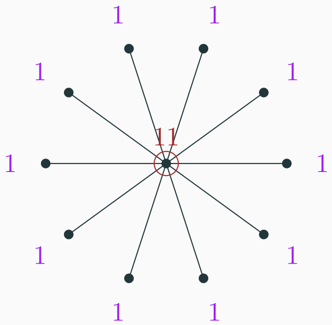

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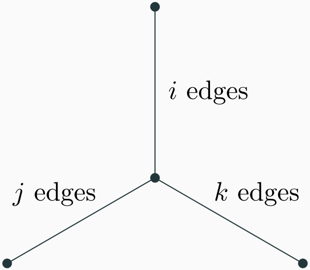

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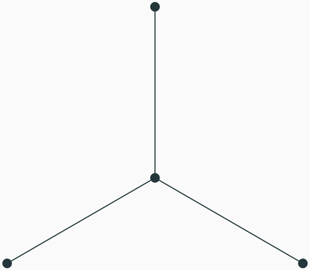

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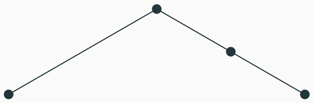

$S_{0,1,2}$

P_n and bull graph

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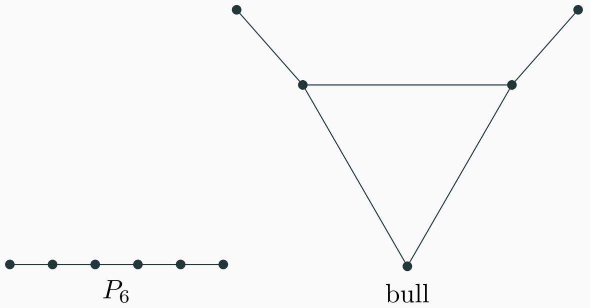

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The MWSS can be solved in time $\mathcal{O}(n^7)$ for every graph on n vertices in the class of $(P_6$, bull)-free graphs.

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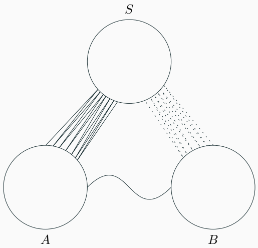

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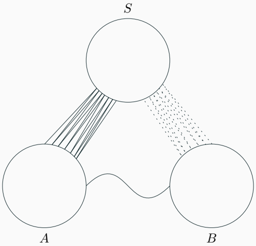

Prime graphs

A graph is prime if every homogeneous set of G is either a singleton or equal to $V(G)$.

Theorem (Lozin, Milanič, 2008)

Let $\mathcal{G}$ be a hereditary class of graphs. If the MWSS problem can be solved in polynomial time for any prime graph G in $\mathcal{G}$, then the MWSS problem can be solved in polynomial time for every graph G in $\mathcal{G}$.

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Reducing the problem

This theorem tells us that we only need to concentrate on prime graphs.

Structure of (P_6, bull)-free prime graphs

Forbidden induced subgraphs

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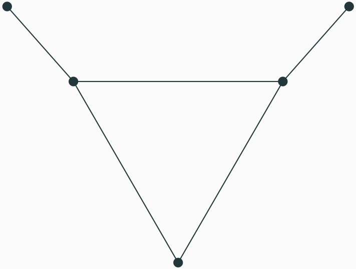
P_6



bull

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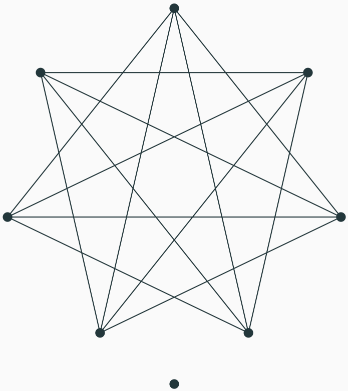

complementary of a 7-wheel

Sketch of the proof

Non-neighborhood approach

Let G be a prime (P_6, bull)-free graph and v a vertex of G.

Let K be the non-neighbors of v.

A MWSS containing v is of weight $w(v) + \alpha_w(G[K])$.

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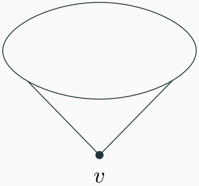

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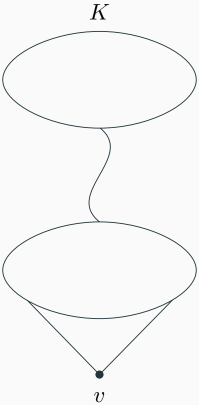

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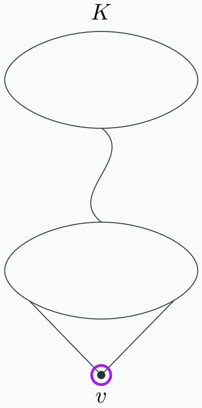

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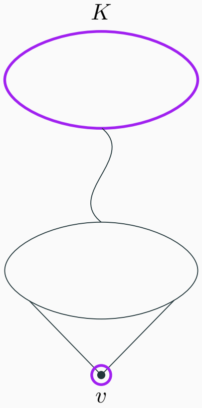

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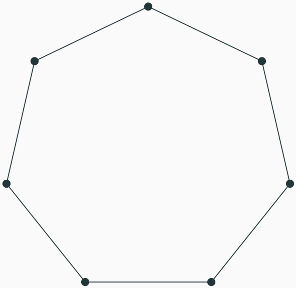

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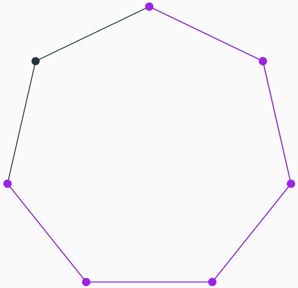

Induces a P_6

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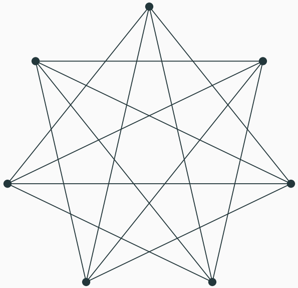

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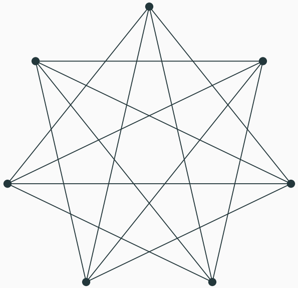

Homogeneous set in G

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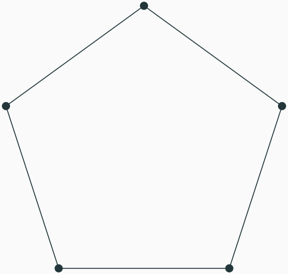

The only possibility is a C_5

Precise structure of the non-neighorhood

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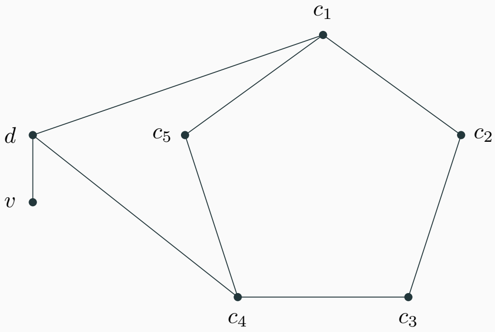

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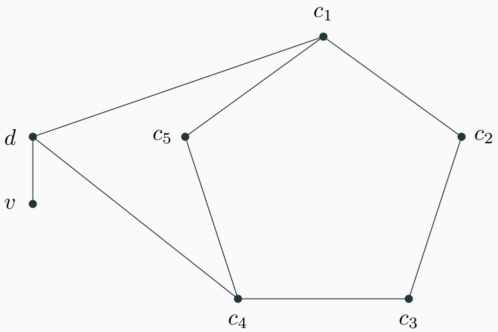

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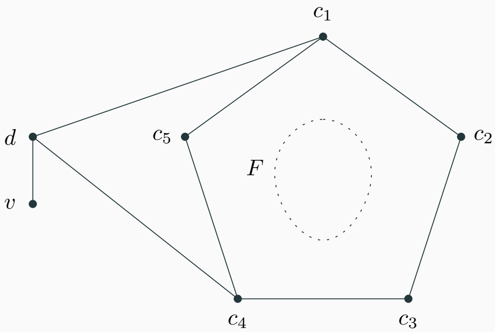

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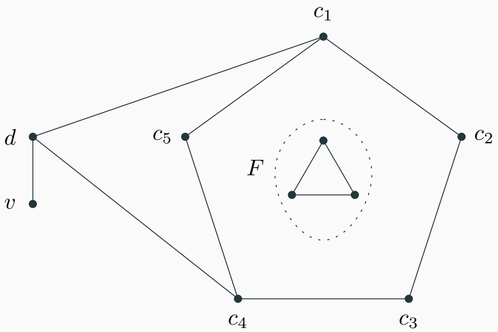

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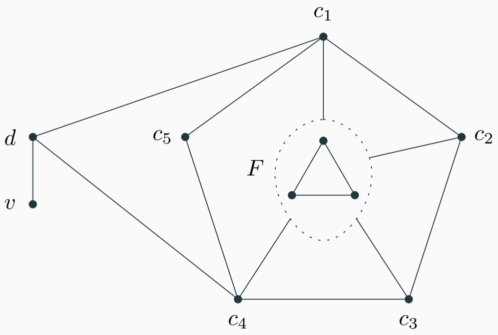

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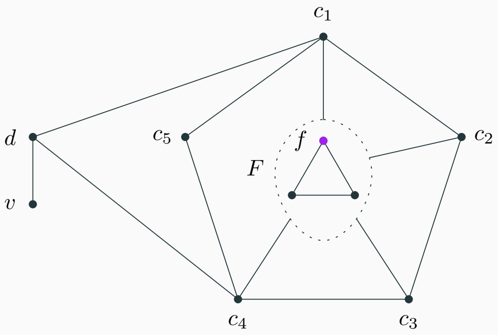

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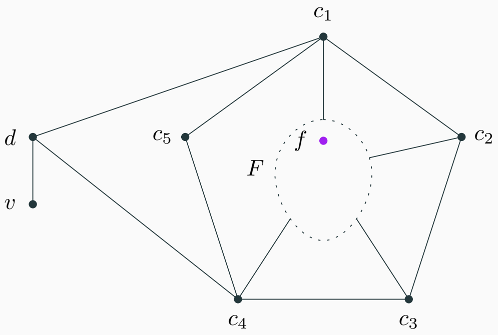

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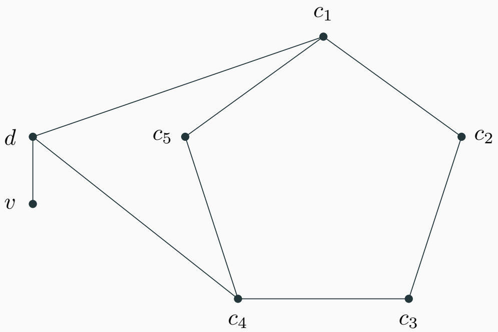

Using the set F to our advantage

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1. for every $f \in F$, compute the MWSS in $K \setminus N(f)$.
2. compute the MWSS in $K \setminus F$.

By the properties of F, in 1. the induced subgraph is **triangle-free**. In 2., since we consider an induced subgraph without the set F, it is also **triangle-free**.

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- Thomassé, Trotignon and Vušković first proved that MWSS is FPT in bull-free graphs. It was later improved by Perret du Cray and Sau. In both papers, the bottleneck against polynomiality is a very precise class called $\mathcal{T}_1$ in Chudnovsky's decomposition theorem. Though, the class $\mathcal{T}_1$ is complicated and not so easy to use for the MWSS problem.

Thank you for your attention.