List-coloring in claw-free perfect graphs

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Joint-work with Sylvain Gravier and Frédéric Maffray

G-SCOP

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Coloring

Given a graph G, a (proper) k-coloring of the vertices of G is a mapping $c : V(G) \to \{1, 2, \ldots, k\}$ for which every pair of adjacent vertices x, y satisfies $c(x) \neq c(y)$.

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Chromatic number

The chromatic number of G, denoted by $\chi(G)$, is the smallest integer k such that G admits a k-coloring.

List-coloring

- Let G be a graph. Every vertex $v \in V(G)$ has a list $L(v)$ of prescribed colors, we want to find a proper vertex-coloring c such that $c(v) \in L(v)$.
- When such a coloring exists, G is L-colorable.

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Choice number

The choice number $ch(G)$ of a graph G is the smallest k such that for every list assignment L of size k, the graph G is L-colorable.

Chromatic inequality

We have $\chi(G) \leq ch(G)$ for every graph G. There are graphs for which $\chi(G) \neq ch(G)$ (in fact, the gap can be arbitrarily large).

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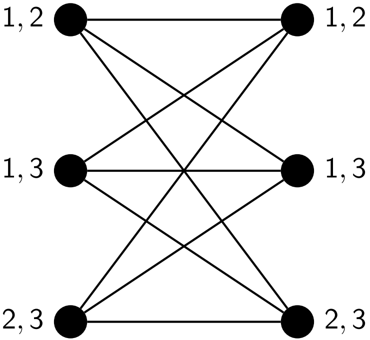

Vizing's conjecture

For every graph G, $\chi(\mathcal{L}(G)) = ch(\mathcal{L}(G))$. In other words, $\chi'(G) = ch'(G)$ with $ch'(G)$ the list chromatic index of G.

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Special case

We are interested in the case where G is perfect.

Perfect graph

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Strong Perfect Graph Theorem

A graph G is perfect if and only if G does not contain an odd hole nor an odd antihole.

Claw-free graph

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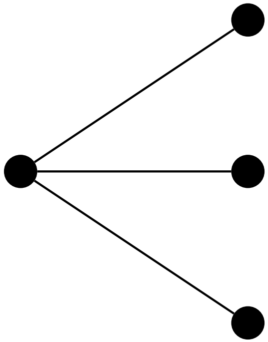
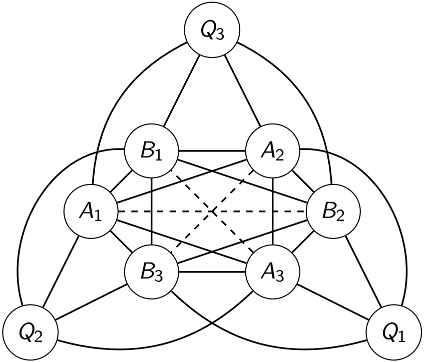
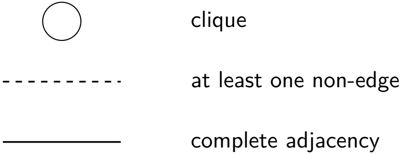


Theorem [Chvátal and Sbihi, 1988]

Every claw-free perfect graph either has a clique-cutset, or is a peculiar graph, or is an elementary graph.


Q_3
B_1
A_2
A_1
B_2
B_3
A_3
Q_2
Q_1


clique
at least one non-edge
complete adjacency

Theorem [Maffray and Reed, 1999]

A graph G is elementary if and only if it is an augmentation of the line-graph H (called the **skeleton** of G) of a bipartite multigraph B (called the **root** graph of G).

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- Add all edges between X and $N_G(x) \setminus \{y\}$ in G'.

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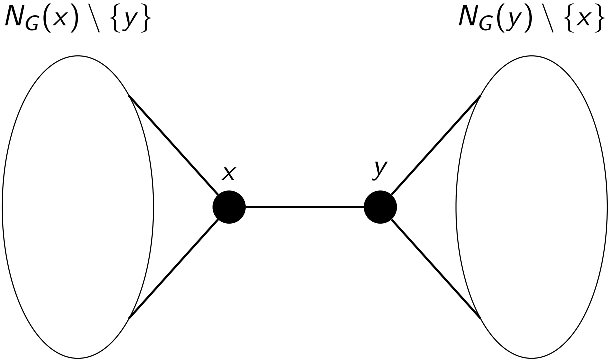
- Add all edges between Y and $N_G(y) \setminus \{x\}$ in G'.


$N_G(x) \setminus \{y\}$
$N_G(y) \setminus \{x\}$
x
y

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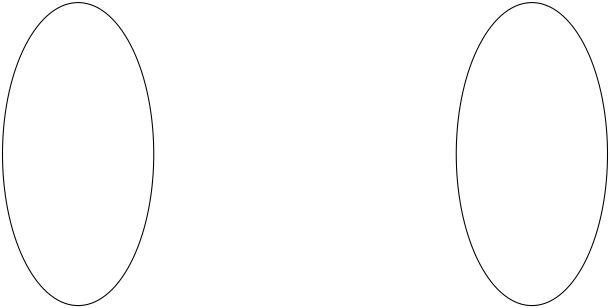

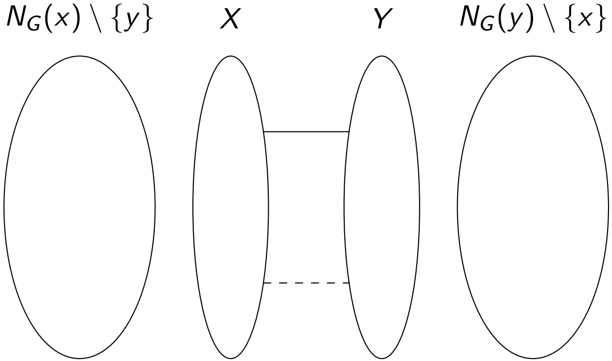
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X
Y
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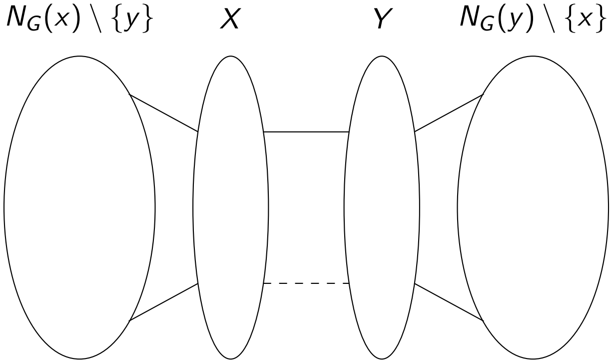
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Theorem [Gravier, Maffray, P.]

Let G be a claw-free perfect graph with $\omega(G) \leq 4$. Then $\chi(G) = ch(G)$.

Lemma

Let G be a connected claw-free perfect graph that contains a peculiar subgraph. Then G is peculiar.

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- Let H be a peculiar proper subgraph of G that is maximal.
- Since G is connected there is a vertex x of $V(G) \setminus V(H)$ having a neighbour in H.
- In order to avoid claws, odd holes and odd anti holes, x has many neighbours in H from several sets of the peculiar partition. In fact, x is in one of those sets, hence $H \cup \{x\}$ is a peculiar graph.

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- If some pairs of non-adjacent vertices share a color, we can color G.
- If no such pair exists, we can find a coloring by Hall's theorem.

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Proof

Several pages.

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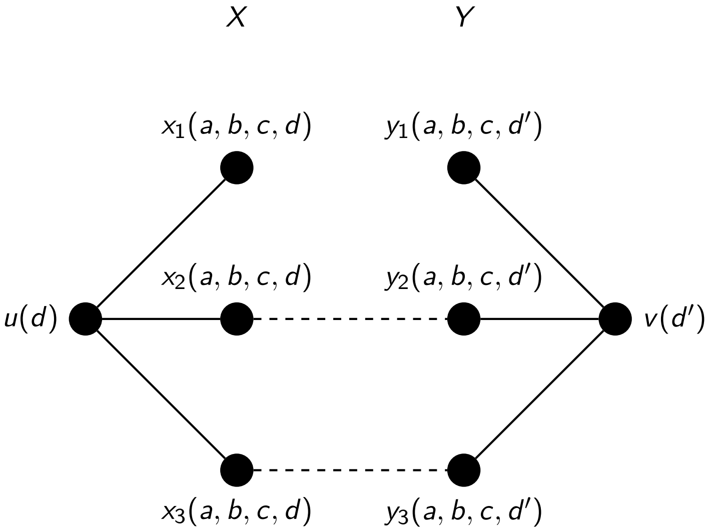
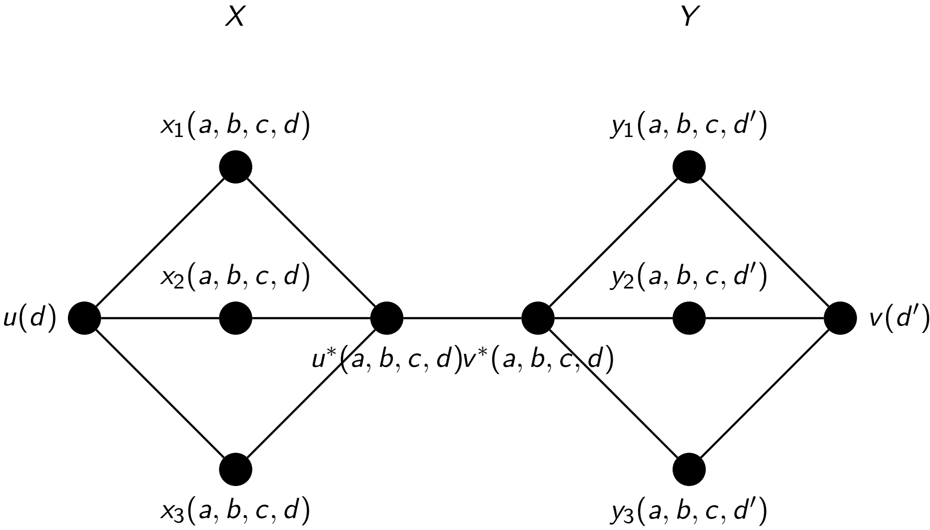
- If the coloring of G' can be extended to (X, Y) we are done.
- If not, we can show thanks to a gadget that there exists a coloring of G' that can be extended to G.


X
Y
x1(a, b, c, d)
y1(a, b, c, d′)
x2(a, b, c, d)
y2(a, b, c, d′)
u(d)
v(d′)
x3(a, b, c, d)
y3(a, b, c, d′)


X
Y
$x_1(a, b, c, d)$
$y_1(a, b, c, d')$
$x_2(a, b, c, d)$
$y_2(a, b, c, d')$
$u(d)$
$v(d')$
$u^*(a, b, c, d)$
$v^*(a, b, c, d)$
$x_3(a, b, c, d)$
$y_3(a, b, c, d')$

Proof of the main theorem

Let G be a claw-free perfect graph and C a clique cutset. The graph $G \setminus C$ has two components A_1 and A_2. Let $G_1 = G[C \cup A_1]$ and $G_2 = G[C \cup A_2]$. We may assume that G_1 is colored and we want to extend it to G_2. Let us assume that G_2 is elementary. There are two cases:

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Proof of 1

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Proof of 2

We use a Galvin argument to show that the graph G_2 is colorable with forced colors on the clique C.

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A word on our method

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- It seems to be hard to use this trick for the general case.

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A word on our method

- Proving that elementary graphs are chromatic-choosable by induction on the number of augmented flat edges gives us interesting tools for the extension of a coloring to an elementary graph.
- It seems to be hard to use this trick for the general case.
- We tried Galvin like arguments without any success.

Thank you for listening.
Do you have any questions?